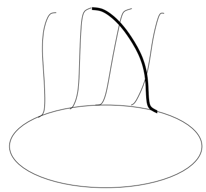
Summer 2014 Geometry / Topology Qualifying Examination

§1. Algebraic Topology

(1) Consider the knot $K \subset S^3$ depicted below. It is realized as a simple closed curve on a *standardly embedded* torus $T \subset S^3$, meaning that $S^3 \setminus T$ consists of two open solid tori.



(a) Choose a basepoint $* \in S^3 \setminus K$ and determine a presentation of $\pi_1(S^3 \setminus K, *)$

§2: Differential Topology

(1) If M is a smooth manifold, show that the tangent bundle TM and the cotangent bundle T^*M are isomorphic. *(Just as with vector spaces, there is no* canonical *isomorphism. You don't have to prove this, though. Also, feel free to assume anything that you like from linear algebra.)*

(2) A *Lie homomorphism* is a smooth homomorphism between Lie groups.

(a) Show that any Lie homomorphism $\varphi : G \to H$ has constant rank: that is, there exists some $k \in \mathbb{Z}$ such that $\operatorname{rank}(d\varphi_g) = k$ for all $g \in G$.

(b) Suppose that G, H are connected n-dimensional Lie groups and $\varphi : G \to H$ is a Lie homomorphism with discrete kernel. Show that φ is a surjective diffeomorphism. *(In fact, φ is a covering map, but the proof of this is homework-level rather than exam-level.)*

(3) Write $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$ and let $\mathcal{G}$ be the pseudogroup generated by all diffeomorphisms φ between open subsets of $\mathbb{R}^n$ that take horizontal factors to horizontal factors: that is,

$$(*) \qquad \varphi(x, y) = (\varphi_1(x, y), \varphi_2(y)),$$

for $x \in \mathbb{R}^{n-k}$ and $y \in \mathbb{R}^k$. Show that $\mathcal{G}$ consists of all diffeomorphisms between open subsets of $\mathbb{R}^n$ whose Jacobian matrix at every point is an $n \times n$ matrix such that the lower left $(n-k) \times k$ block is 0. *(Showing that the set of diffeomorphisms satisfying the Jacobian property is a pseudo-group is almost immediate, although you should at least say what the properties are. The real point here is to explain why it is the* minimal *pseudo-group containing all such φ.)*

A $\mathcal{G}$-structure on an n-manifold M is called a *codimension k foliation* of M. Since at least locally, the transition maps preserve the decomposition of $\mathbb{R}^n$ into horizontal slices, these slices piece together to give a decomposition of M into submanifolds, called the *leaves* of the foliation.

(4) Show that the antipodal map $A : S^n \to S^n$, $A(x) = -x$ is homotopic to the identity if and only if n is odd. *(Feel free to use Lefschetz theory if you would like.)*

(5) Show that a closed 1-form ω on a manifold M is exact if and only if $\int_{S^1} f^*\omega = 0$ for every smooth map $f : S^1 \to M$. *(Feel free to use Stokes' theorem, but you shouldn't reference deRham cohomology or anything that implicitly relies on this result.)*