

# Real Analysis Qual

June 10, 2016

**Problem 1.** Let  $\sim$  be the equivalence relation on the interval  $[0, 1]$  given by  $x \sim y \iff x - y \in \mathbb{Q}$ . Choose one element from each equivalence class (using the Axiom of Choice). Let  $A \subset [0, 1]$  denote the set of these chosen elements. For a given set  $B \subset \mathbb{R}$ , define  $B + x = \{y + x \mid y \in B\}$ .

- (a) Show that the sets  $A + q$ , for  $q \in \mathbb{Q} \cap [-1, 1]$ , are disjoint.
- (b) Show that  $[0, 1] \subset \bigcup_{q \in \mathbb{Q} \cap [-1, 1]} (A + q) \subset [-1, 2]$ .
- (c) Show that  $A$

(c) Prove Fatou's Lemma, which states that

$$\liminf_n \int f_n \leq \int \liminf_n f_n$$

for any sequence  $(f_n)$  of non-negative, measurable functions. To get started, let  $g_n = \inf_{i \geq n} f_i$ . Observe that

$E_1, \dots, E_n \subset A$  and  $c_1, \dots, c_n \in \mathbb{R}_0^+$ . Define  $\nu : A \rightarrow [0, \infty]$  by  $\nu(X) < \infty$ . Fix

$$\nu(A) = \sum_{i=1}^n c_i \mu(A \cap E_i).$$

- (a) Show that  $\nu$  is a measure.
- (b) Show that  $\nu$  is absolutely continuous with respect to  $\mu$ .



## Complex Analysis Qualifying Exam – Spring 2016

Please answer the following problems. Explain your argument carefully – if you refer to a well-known theorem from class, please state the theorem precisely and explain why it applies.

Notation:  $D$  = open unit disk,  $\mathbb{C}^* = \mathbb{C} - \{0\}$ ,  $H$  = upper half plane.

- 1) Find a holomorphic function  $f$  on  $\mathbb{C}^*$  such that
  - $f$  is a pointwise limit of polynomial functions, but
  - $f$  is not a uniform limit of polynomial functions (that is, there is no sequence of polynomials that converges to  $f$  uniformly on compact subsets of  $\mathbb{C}^*$ ).

Prove both assertions for your choice of  $f$ .

- 2) Find a biholomorphism between  $H$  and the region

$$U = \{z \in \mathbb{C} \mid |z - 1| < 1, |z - i| < 1\}.$$

It is enough to write down explicitly functions whose composition yields a biholomorphism from  $H$  to  $U$  or from  $U$  to  $H$ .

- 3) Let  $U \subset \mathbb{C}$  be a simply connected region. For any point  $a \in U$ , the Green function of  $U$

where  $\zeta(z)$  is the Weierstrass  $\zeta$ -function

$$\zeta(z) = \frac{1}{z^2} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( \frac{1}{(z-n)^2} - \frac{1}{n^2} \right).$$

Justify carefully any techniques you use.