

$$X = (x_1, s_1; \dots; x_n)$$



(3)

$$X \succeq Y \iff V^*(c_X, p_1, \dots, p_n) \geq V^*(c_Y, p_1, \dots, p_n)$$

**Lemma 1.** Let  $\succeq$  be a distribution-regret preference relation. Then  $\succeq$  admits a two-dimensional regret function  $V^* : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{R}$  and a regret functional  $V^*$  such that

$$\begin{aligned} X \succeq Y &\iff V^*(c_X, p_1; \dots; c_Y, p_n) \geq 0 \\ &\iff V^*(c_Y, q_1; \dots; c_X, q_m) \leq 0 \end{aligned}$$

where  $c_X$  and  $c_Y$  are the certainty equivalents of  $X$  and  $Y$  respectively.

**Proof.** Let  $V^*$  be a regret functional such that  $V^*(x, y) \geq 0$  for all  $x, y \in \mathcal{D}$ ,

$$V^*(x, y) = V^*(x, \delta_y) - V^*(\delta_y, y) \quad \text{for } y \in \mathcal{D},$$

$$V^*(c_X, p_1; \dots; c_Y, p_n) = V^*(c_X, \delta_{c_Y}) - V^*(\delta_{c_Y}, c_Y)$$

$$\begin{aligned} X \succeq Y &\iff X \succeq \delta_{c_Y} \\ &\iff V^*(c_X, \delta_{c_Y}) - V^*(\delta_{c_Y}, c_X) \geq 0 \\ &\iff V^*(c_X, p_1; \dots; c_Y, p_n) \geq 0 \quad \square \end{aligned}$$

**Definition 4.**  $\succeq$  distribution-regret based

$$: \mathcal{D} \times \mathcal{D} \rightarrow \mathfrak{R},$$

$$V : \rightarrow \mathfrak{R},$$

$$X \succeq Y \text{ iff } V(\Psi(X, c_Y)) \geq 0 \text{ iff } 0 \geq V(\Psi(Y, c_X)),$$

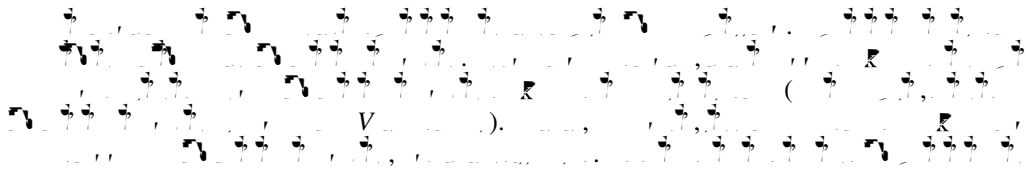
$$\Psi(X, c_Y) = ( (x_1, c_Y), p_1; \dots; (x_n, c_Y), p_n)$$

$$X \delta_{c_Y} (Y), \Psi(Y, c_X)$$

$$Y \delta_{c_X} (X),$$

x

$$X \sim \delta_{c_Y} \implies V(n d \quad V$$



**Proposition 3.** *If the preference relation  $\succeq$  is consistent then it satisfies distribution regret.*

**Proof.** Let  $Z \in \mathcal{L}$  and  $Z = \delta_x$  for some  $x \in \mathcal{D}$ . Then  $f(c_Z, \lambda) = 0$ .<sup>10</sup> Let  $\lambda(Z) = \lambda(Y)$ . Then  $f(x, c_Y) = f(x, \lambda(Y))$ .<sup>11</sup> Since  $\succeq$  is consistent,  $U(\delta_x) = x$  and  $f(x, c_Y) = f(x, \lambda(Y)) = 0$ . Thus  $f(x, \lambda(Y)) \in \mathcal{D}$ .

$$\begin{aligned} f(X, \lambda(Y)) &= (f(x_1, \lambda(Y)), p_1; \dots; f(x_n, \lambda(Y)), p_n) \\ &= ((x_1, c_Y), p_1; \dots; (x_n, c_Y), p_n) \\ &= \Psi(X, c_Y) \end{aligned} \tag{4}$$

$$\begin{aligned} X \succeq Y \sim \delta_{c_Y} &\iff f(X, \lambda(Y)) \succeq f(Y, \lambda(Y)) \sim \delta_{f(c_Y, \lambda(Y))} = \delta_0 \\ &\iff U(f(X, \lambda(Y))) \geq U(f(Y, \lambda(Y))) = U(\delta_0) = 0 \end{aligned}$$

$$V(\Psi(X, c_Y)) = U(f(X, \lambda(Y))) = U(\Psi(X, c_Y))$$

$$\begin{aligned} X \succeq Y &\iff U(\Psi(X, c_Y)) \geq 0 \\ &\iff V(\Psi(X, c_Y)) \geq 0 \end{aligned}$$

□

Let  $Z = (z_1, r_1; \dots; z_n, r_n)$  with  $z_1 \leq \dots \leq z_n$ . Then  $f(c_Z, \lambda(Z)) = 0$ .<sup>12</sup> Let  $\lambda(Z) = \lambda(Y)$ . Then  $f(x, \lambda(Y)) = u^{-1}(u(x) + \lambda)$ .

$$\begin{aligned} f(c_Z, \lambda(Z)) = 0 &\implies \lambda(Z) = -u(z_1)g(r_1) - \sum_{i=2}^n u(z_i) \left[ g\left(\sum_{j=1}^i p_j\right) - g\left(\sum_{j=1}^{i-1} p_j\right) \right] \end{aligned}$$

Since  $U(Z) = c_Z$ , we have  $U(\delta_x) = x$  and  $f(x, c_Y) = f(x, \lambda(Y)) = 0$ .

<sup>10</sup>  $0 \in [\mathcal{D}]$ . For any  $d \in [\mathcal{D}]$ ,  $f(c_Z, \lambda(Z)) = d$  if and only if  $(x, x) = d$ .  
<sup>11</sup>  $(x, x) = (x, c_{\delta_x}) = f(x, \lambda(\delta_x)) = 0$ .



$$(x, c_Y) = f(x, \lambda(Y)) = u^{-1}(u(x) + \lambda(Y))$$



**Example 3.**

$\succsim$  is a preference relation on  $\mathcal{I}$ ,  $\lambda$  is a linear functional on  $\mathcal{I}$ ,  $\lambda(X) = 1$ ,  $\lambda(Y) = 0$ ,  $\lambda(Z) = 0$ ,  $\lambda(X') = 1$ ,  $\lambda(Y') = 0$ ,  $\lambda(Z') = 0$ .

$X, Y, Z \in \mathcal{I}$ .  $V(\Psi(X, c_Z)) = V(\Psi(Y, c_Z)) = 0$ ,  $\alpha \in (0, 1)$ ,  $V(\Psi(\alpha X + (1 - \alpha)Y, c_Z)) = 0$ .

$$\Psi(X', c_{Z'}) =$$

$$[X] + \alpha \mu_X^+ = (1 + \alpha)\alpha \implies \alpha = \frac{-(1 - \mu_X^+) + \sqrt{(1 - \mu_X^+)^2 + 4 [X]}}{2} \quad (9)$$

$$[X] > 0 \implies (X < 0) > 0 \implies X > \delta_{[X]} \quad (10)$$

Since  $\mu_X^+ > E[X] > 0$ ,  $X \sim \delta_\alpha$ ,  $\alpha > E[X]$ .  
 $f(-1, \lambda_0) = s$ ,  $f(t, \lambda_0) = 0$ ,  $z \geq t$ ,  $-1 < s, t < 0$ ,  $\lambda_0$

$$\left[ \left( z, \frac{1+t}{1+z}; -1, \frac{z-t}{1+z} \right) \right] = t, \quad \left( z, \frac{1+t}{1+z}; -1, \frac{z-t}{1+z} \right) \sim (t, 1)$$

$$\left( f(z, \lambda_0), \frac{1+t}{1+z}; s, \frac{z-t}{1+z} \right) \sim (0, 1) \quad (10)$$

**Proposition 4.** *If the preference relation  $\succeq$  satisfies distribution regret with a commutative regret function  $f$ , then it is consistent.*

**Proof.**  $\forall d \in \mathbb{R}$ ,  $(x, x) = d$ ,  $x \in \mathcal{D}$ . (4)  $f(x, \lambda) = y$ ,  $(x, y) = d - \lambda$ .  
 $(x, f(x, \lambda)) = d - \lambda$  (13)

$X \succeq Y$ ,  $V(\Psi(X, c_Y)) \geq 0$ . (13)

$$(c_X, f(c_X, \lambda)) = (c_Y, f(c_Y, \lambda)) = d - \lambda$$

$$(12) \quad (c_X, c_Y) = (f(c_X, \lambda), f(c_Y, \lambda)),$$

$$\Psi(\delta_{c_X}, c_Y) = \Psi(\delta_{f(c_X, \lambda)}, f(c_Y, \lambda))$$

$$\delta_{c_X} \geq \delta_{c_Y} \quad (13) \quad \delta_{f(c_X, \lambda)} \geq \delta_{f(c_Y, \lambda)}. \quad (12)$$

$$(x_i, c_X) = (f(x_i, \lambda), f(c_X, \lambda))$$

$$\Psi(X, c$$

### 5. Discussion

only

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