A Reconsideration of Money Growth Rules

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Abstract

A New Keynesian model, estimated using Bayesian methods over a sample period that includes the recent episode of zero nominal interest rates, illustrates the e ects of replacing the Federal Reserve's historical policy of interest rate management with one targeting money growth instead. Counterfactual simulations show that a rule for adjusting the money growth rate, modestly and gradually, in response to changes in the output gap delivers performance comparable to the estimated interest rate rule in stabilizing output and in ation. The simulations also reveal that, under the same money growth rule, the US economy would have recovered more quickly from the 2007-09 recession, with a much shorter period of exceptionally low interest rates. These results suggest that money growth rules can serve as a simple and e ective alternative guide for monetary policy in the current low interest rate environment.

JEL Codes: E31, E32, E41, E47, E51, E52.

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1 Introduction

For the past quarter century, and perhaps longer, the Federal Reserve has conducted monetary policy by managing nominal interest rates. While today's practice of strict federal funds rate targeting has its origins in the early 1990s, Greenspan (1997), Meulendyke (1998), and Thornton (2007) all describe Federal Reserve policy as shifting towards tighter interest rate control beginning sometime in the 1980s. Cook (1989) goes back even further, arguing that the reserves targeting procedures used from 1979 through 1982 disguised policy actions taken to manage the funds rate instead.

Academic economists also depict Federal Reserve policy as managing interest rates. Taylor (1993) introduced his now-famous rule, which describes how the Fed adjusts its interest rate target in response to movements in the output gap and inflation. Taylor (1993) also demonstrates that the strikingly simple formula tracks actual movements in the federal funds rate remarkably well over the period from 1987 through 1992. Some variant of the Taylor rule now appears as the description of monetary policy in textbook New Keynesian models presented, for example, by Woodford (2003) and Galí (2015).

Preference for interest rate management, in both practice and theory, often is motivated with reference to Poole's (1970) classic analysis, demonstrating that in a stochastic IS-LM model, policies targeting the nominal interest rate insulate output from the effects of money demand shocks, whereas policies targeting the money stock instead allow these shocks to contribute to macroeconomic volatility. Poole's model holds the aggregate price level fixed, but Ireland (2000), Collard and Dellas (2005), and Galí (2015) demonstrate that these results extend to modern New Keynesian models as well, in which monetary policies calling for a constant rate of money growth lead to excess volatility in both output and inflation, compared to policies targeting interest rates instead, especially when the economy is hit by recurrent money demand shocks. Furthermore, as emphasized by Ireland (2004) and Belongia and Ireland (2018), standard New Keynesian models feature forward-looking variants of more traditional Keynesian IS and Phillips curve that imply monetary policy affects out-

of these studies considers the possibility that money growth rules might work significantly better if they allowed policy to adjust to movements in the output gap and inflation in a manner similar to that of the Taylor rule.

Thus, this paper extends previous work by reconsidering money growth rules in an estimated New Keynesian model. By identifying a parsimonious rule that dictates a systematic response of money growth to changes in the output gap, it follows in the same style of research presented, for instance, in Taylor (1999) by characterizing rules that remain simple while still delivering favorable economic outcomes. And by using counterfactual simulations to assess how the US economy would have performed over a sample period running from 1983 through 2018, it illustrates the satisfactory performance of a money growth rule in both good times – the period of the Great Moderation – and bad – the Great Recession and its aftermath.

The particular variant of the New Keynesian model used here takes most of its basic features from those in Ireland (2004a, 2004b, 2007, 2011), but innovates in four distinct ways. First, it introduces real money balances into a representative household's utility function in a manner that leaves the New Keynesian IS and Phillips curves in their standard forms, excluding the additional terms involving money growth that appear in Ireland (2004b). This ensures that the extended model retains the New Keynesian assumption that monetary policy actions have an impact on output and inflation only through their effects on the current and expected future path of the short-term nominal interest rate. The intent is to put money growth rules to a most stringent test, by excluding model features that might specifically favor stability in the money stock.

Second, the model's money-in-the-utility function specification is also tailored to imply that the level of real balances demanded by the non-bank public remains finite even as nominal interest rates fall to zero, reflecting observations made by Ireland (2009) and Rognlie (2016) that US money demand did not explode during either episode of very low nominal interest rates following the last two recessions. Intriguingly, as noted by Roglie (2016), this specification implies that short-term interest rates can fall below zero, at least by modest amounts for short periods of time, in a well-defined equilibrium – a phenomenon that will be explored in the counterfactual experiments performed with the estimated model.[1] Third, the model includes adjustment costs of real balances in its specification, following Nelson (2002) and Andrés, López-Salido, and Nelson (2004, 2009), all of which present evidence that New Keynesian models with money fit the data better when they allow for gradual adjustment of real balances to shocks that hit the economy.

Fourth and finally, the analysis here employs methods developed by Kulish, Morley, and Robinson (2017) to account for periods, like that experienced in the US from 2009 through 2015, when short-term nominal interest rates were constrained by the central bank to remain near zero. According to the New Keynesian model, even after its current policy rate is lowered to zero, the central bank can use "forward guidance," in the form of policy announcements that lengthen private agents' expectations regarding the duration of the zero interest rate episode, to deliver additional monetary stimulus. The Bayesian estimation methods used here exploit survey data to track changes in the expected duration of the zero interest rate period and the effects these shifts in expectations have on output and inflation. Thus, with these methods, the model can be estimated over a sample running continuously from 1983 through 2018, accounting for the effects of both zero interest rates and forward guidance over the 2009-15 period as well as the effects of more traditional interest rate policy before and after. The estimated model can then be used to explore counterfactual scenarios in which the central bank systematically adjusts its target for the money growth rate under both favorable and unfavorable economic conditions.

The results from this exercise reveal that, even in a model that departs minimally from standard New Keynesian specifications and therefore offers no special role for changes in

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the money stock, a money growth rule nonetheless can deliver performance on par with that generated by more conventional Taylor rules for the interest rate. The counterfactual simulations show, in particular, that under a money growth rule that responds modestly but persistently to changes in the output gap, the US economy would have recovered more quickly than it actually did from the financial crisis and Great Recession, without requiring a prolonged period of zero or negative interest rates. Thus, the results suggest that as Federal Reserve officials search for a new policy framework within which they can more reliably achieve their stabilization objectives in an environment of low interest rates and inflation following a series of adverse disturbances, abandoning the traditional practice of managing the federal funds rate in favor of a rule targeting the money growth rate should be added to the list of possibilities considered.

2 The Model

2.1 Overview

The model economy consists of a representative household, a representative finished goods-producing firm, a continuum of intermediate goods-producing firms indexed by $i \in [0, 1]$, and a central bank. During each period $t = 0, 1, 2, \ldots$, each intermediate goods-producing firm produces a distinct intermediate good. Hence, intermediate goods are also indexed by $i \in [0, 1]$, with good i produced by firm i. The model features enough symmetry, however, to allow the analysis to focus on the behavior of a representative intermediate goods-producing firm that manufactures the generic intermediate good.

Habit formation introduced through the representative household's preferences and incomplete indexation of sticky nominal goods prices set by monopolistically competitive intermediate goods-producing firms imply that the model's New Keynesian IS and Phillips curves include both backward and forward-looking elements. The estimation procedure allows the data to decide on the relative importance of these backward and forward-looking terms. The central bank in the estimated model conducts monetary policy according to a version of the Taylor (1993) rule, reflecting the Federal Reserve's actual practice of federal funds rate targeting over most if not all of the 1983-2018 sample period.[2] As noted above, however, the introduction of a money demand curve of a form that is consistent with the same US data also permits consideration of counterfactual monetary policy rules for money growth targeting instead.

2.2 The Representative Household

The representative household enters each period $t = 0, 1, 2, \ldots$ with M_{t-1} units of money and B_{t-1} bonds. At the beginning of period t, the household receives a lump-sum monetary transfer T_t from the central bank. In addition, the household's bonds mature, yielding B_{t-1} additional units of money. The household uses some of this money to purchase B_t new bonds at the price of $1/r_t$ units of money per bond; thus, r_t denotes the gross nominal interest rate between t and $t+1$.

During period t, the household supplies $h_t(i)$ units of labor to each intermediate goods-producing firm $i \in [0, 1]$. The household gets paid at the nominal wage rate W_t, earning $W_t h_t$ i[illegible]bor income, where

$$h_t = \int_0^1 h_t(i)\, di$$

deno[illegible]tal hours worked during the period. Also during period t, the household consumes C_t u[illegible]f the finished good, purchased at the nominal price P_t from the representative finish[illegible]ods-producing firm.

A[illegible]end of period t, the household receives nominal profits $D_t(i)$ from each intermediate good[illegible]ducing firm $i \in [0, 1]$. The household then carries M_t units of money into period

[2] Th[illegible]eral Reserve has never announced an explicit rule to guide the setting of its interest rate target. Nevert[illegible]alysis here adopts the assumption made throughout the literature on New Keynesian econom[illegible]pirically, changes in the federal funds rate target can be described accurately by a rule of the form originally proposed by Taylor (1993). Belongia and Ireland (2019

$t+1$, chosen subject to the budget constraint

$$\frac{M_{t-1} + T_t + B_{t-1} + W_t h_t + D_t}{P_t} \geq C_t + \frac{M_t + B_t/r_t}{P_t} \tag{1}$$

for all $t = 0, 1, 2, \ldots$, where

$$D_t = \int_0^1 D_t(i)\, di$$

denotes total profits received for the period.

The household's preferences are described by the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t a_t \left[\ln(C_t - \gamma C_{t-1}) + v\left(\frac{M_t}{P_t Z_t}; u_t\right) - \frac{\phi_m}{2}\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)^2 \frac{M_t}{P_t Z_t} - \eta h_t \right],$$

where both the discount factor and the habit formation parameter lie between zero and one, with $0 < \beta < 1$ and $0 \leq \gamma \leq 1$. The preference shock a_t follows the stationary autoregressive process

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at} \tag{2}$$

for all $t = 0, 1, 2, \ldots$, with $0 \leq \rho_a < 1$, where the serially uncorrelated innovation ε_{at} is normally distributed with mean zero and standard deviation σ_a. Utility is additively distributed with mean zero and standard deviation σ_z. The shock u_t to money demand follows the stationary autoregressive process

$$\ln(u_t) = \rho_u \ln(u_{t-1}) + \varepsilon_{ut} \tag{4}$$

for all $t = 0, 1, 2, \ldots$, with $0 \le \rho_u < 1$, where the serially uncorrelated innovation ε_{ut} is normally distributed with mean zero and standard deviation σ_u. Finally, the parameter $\phi_m \ge 0$ governs the magnitude of the adjustment cost for real balances, adapted from Nelson (2002) and Andrés, López-Salido, and Nelson (2004, 2009) to take the quadratic functional form used here. Since these costs subtract from utility along with hours worked, they have the interpretation as a time cost, and are scaled by the average growth rate parameter z from (3) so as to equal zero in the model's steady state.

Thus, the household chooses C_t, h_t, B_t, and M_t for all $t = 0, 1, 2, \ldots$ to maximize expected utility subject to the budget constraint (1) for all $t = 0, 1, 2, \ldots$. The first-order conditions for this problem can be written as

$$\Lambda_t = \frac{a_t}{C_t - \gamma C_{t-1}} - \beta\gamma E_t\left(\frac{a_{t+1}}{C_{t+1} - \gamma C_t}\right); \tag{5}$$

$$a_t = \Lambda_t (W_t/P_t); \tag{6}$$

$$\Lambda_t = \beta r_t E_t(\Lambda_{t+1}/\Pi_{t+1}); \tag{7}$$

$$\begin{aligned} & a_t v_1\left(\frac{M_t}{P_t Z_t}; u_t\right) - a_t\left(\frac{\phi_m}{2}\right)\left[\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right]^2 \\ & - a_t \phi_m \left[\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right]\left[\frac{M_t/P_t}{zM_{t-1}/P_{t-1}}\right] \\ & + \beta\phi_m E_t\left\{a_{t+1}\left[\frac{M_{t+1}/P_{t+1}}{zM_t/P_t} - 1\right]\left[\frac{M_{t+1}/P_{t+1}}{zM_t/P_t}\right]^2\left(\frac{zZ_t}{Z_{t+1}}\right)\right\} \\ & = Z_t\Lambda_t\left(1 - \frac{1}{r_t}\right); \end{aligned} \tag{8}$$

and (1) with equality for all $t = 0, 1, 2, \ldots$, where Λ_t denotes the nonnegative Lagrange multiplier on the budget constraint for period t, $_t = P_t{=}P_{t\ \ 1}$ denotes the gross in ation rate between t and $t+1$, and v_1 denotes the partial derivative of the function v with respect to its rst argument, scaled real balances.

In the special case where

becomes

$$
\begin{aligned}
&\frac{a_t}{} \ln(m\ ) \quad \ln \frac{M_t}{P_t Z_t} + \ln(u_t) \quad a_t \ \frac{m}{2} \quad \frac{M_t{=}P_t}{zM_{t\ \ 1}{=}P_{t\ \ 1}} \quad 1 \ ^2 \\
&a_t \ \ m \quad \frac{M_t{=}P_t}{zM_{t\ \ 1}{=}P_{t\ \ 1}} \quad 1 \quad \frac{M_t{=}P_t}{zM_{t\ \ 1}{=}P_{t\ \ 1}} \qquad \# \\
&+ \quad {}_m E_t \ \ a_{t+1} \quad \frac{M_{t+1}{=}P_{t+1}}{zM_t{=}P_t} \quad 1 \quad \frac{M_{t+1}{=}P_{t+1}}{zM_t{=}P_t} \ ^2 \quad \frac{zZ_t}{Z_{t+1}} \\
&= Z_t \ \ {}_t \quad 1 \quad \frac{1}{r_t} \quad ;
\end{aligned}
\tag{9}
$$

2.3 The Representative Finished Goods-Producing Firm

During each period $t = 0; 1; 2; :::$, the representative nished goods-producing rm uses $Y_t(i)$ units of each intermediate good i 2 $[0; 1]$, purchased at the nominal price $P_t(i)$, to manufacture Y_t units of the nished good according to the technology described by

$$
Z_{\ 0}^{\ 1} Y_t(i)^{(\ _t \quad 1) = \ _t} di \qquad ^{_t = (\ _t \quad 1)} \qquad Y_t ;
$$

where $_t$ translates into a random shock to the intermediate goods-producing rms' desired markup of price over marginal cost and therefore acts like a cost push shock of the kind introduced into the New Keynesian model by Clarida, Gal, and Gertler (1999). Here, this markup shock follows the stationary autoregressive process

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:04 5.9776 Tf 8.185t

$$\ln(\ _t i \ _t \ \ _t$$

The rst-order conditions for this problem are

$$Y_t(i) = [\ P_t(i)=P_t]^{\quad t}\, Y_t$$

for all i 2 $[0;1]$ and $t = 0;1;2;:::$.

Competition drives the nished goods-producing rm's pro ts to zero in equilibrium, determining P_t as

$$P_t = \int_0^1 P_t(i)^{1\quad t}\, di^{\quad 1=(1\quad t)}$$

for all $t = 0;1;2;:::$.

2.4 The Representative Intermediate Goods-Producing Firm

During each period $t = 0;1;2;:::$, the representative intermediate goods-producing rm hires $h_t(i)$ units of labor from the representative household to manufacture $Y_t(i)$ units of intermediate good i according to the technology described by

$$Z_t h_t(i) \quad Y_t(i); \tag{11}$$

where Z_t is the aggregate productivity shock introduced in (3).

Since the intermediate goods substitute imperfectly for one another in producing the nished good, the representative intermediate goods-producing rm sells its output in a monopolistically competitive market, setting its nominal price $P_t(i)$ subject to the requirement that it satisfy the representative nished goods-producing rm's demand at that price. Following Rotemberg (1982), the intermediate goods-producing rm faces a quadratic cost of adjusting its nominal price between periods, measured in terms of the nished good and given by

$$\frac{\ _p}{2} \quad \frac{P_t(i)}{_{t\ 1}\ ^1\ P_{t\ 1}(i)} \quad 1^{\ 2}\ Y_t;$$

where $_p$ 0 governs the magnitude of the price adjustment cost, is a parameter that lies between zero and one, with 0 1, and denotes the steady-state rate of inflation. According to this specification, the extent to which price setting is backward-looking depends on the magnitude of the parameter . When, in particular, = 1, prices are indexed fully to past inflation, giving price setting an important backward-looking component. At the other extreme however, when = 0, there is no indexation of prices to past inflation rates and price setting is purely forward-looking.

The cost of price adjustment makes the intermediate goods-producing firm's problem dynamic: it chooses $P_t(i)$ for all $t = 0; 1; 2; \ldots$ to maximize its total real market value, proportional to

$$E_0 \sum_{t=0}^{\infty} {}^{t} {}_{t}[D_t(i)/P_t];$$

where ${}^{t}\,{}_{t}$ measures the marginal utility value to the representative household of an additional unit of real profits received in the form of dividends during period t and where

$$\frac{D_t(i)}{P_t} = \left(\frac{P_t(i)}{P_t}\right)^{1\ \ t} Y_t \quad \left(\frac{P_t(i)}{P_t}\right)^{t} \frac{W_t}{P_t} \quad \frac{Y_t}{Z_t} \quad \frac{}{2}\left(\frac{P_t(i)}{{}_{t\ 1}^{\ \ 1}\ P_{t\ 1}(i)} \quad 1\right)^2 Y_t \qquad (12)$$

measures the firm's real profits during the same period t. The first-order conditions for this problem are

$$0 = (1 \quad {}_{t})\left(\frac{P_t(i)}{P_t}\right)^{t} + {}_{t} \quad P_t(i)$$

2.5 The Efficient Level of Output and the Output Gap

A social planner for this economy who can overcome the frictions associated with monetary trade, sluggish price adjustment, and the monopolistically competitive structure of the intermediate goods-producing sector chooses Q_t and $n_t(i)$ for all $i \in [0,1]$ to maximize the social welfare function

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[a_t \ln(Q_t - \gamma Q_{t-1}) - \int_0^1 n_t(i)\, di \right]$$

subject to the aggregate feasibility constraint

$$Z_t \left[\int_0^1 n_t(i)^{(\theta_t - 1)/\theta_t}\, di \right]^{\theta_t/(\theta_t - 1)} \geq Q_t$$

for all $t = 0, 1, 2, \ldots$. The first-order conditions for this problem are

$$\xi_t = \frac{a_t}{Q_t - \gamma Q_{t-1}} - \beta\gamma E_t \left(\frac{a_{t+1}}{Q_{t+1} - \gamma Q_t} \right),$$

$$a_t = \xi_t Z_t (Q_t/Z_t)^{1/\theta_t} n_t(i)^{-1/\theta_t}$$

for all $i \in [0,1]$, and the aggregate feasibility constraint with equality for all $t = 0, 1, 2, \ldots$, where ξ_t denotes the nonnegative Lagrange multiplier on the aggregate feasibility constraint for period t.

The second optimality condition listed above implies that $n_t(i) = n_t$ for all $i \in [0,1]$ and $t = 0, 1, 2, \ldots$, where

$$n_t = (\xi_t/a_t)^{\theta_t} Z_t^{\theta_t} (Q_t/Z_t).$$

Substituting this last relationship into the aggregate feasibility constraint yields

$$\xi_t = a_t/Z_t.$$

for all $t = 0; 1; 2; \ldots$. To help keep track of the model's observable variables, it is useful to let

$$g_t = Y_t / Y_{t-1} \qquad (18)$$

denote the growth rate of output for all $t = 0; 1; 2; \ldots$.

2.7 Symmetric Equilibrium

In a symmetric equilibrium, all intermediate goods-producing firms make identical decisions, so that $Y_t(i) = Y_t$, $h_t(i) = h_t$, $D_t(i) = D_t$, and $P_t(i) = P_t$ for all $i \in [0; 1]$ and $t = 0; 1; 2; \ldots$. In addition, the market clearing conditions $M_t = M_{t-1} + T_t$ and $B_t = B_{t-1} = 0$ for money and bonds must hold for all $t = 0; 1; 2; \ldots$. After imposing these equilibrium conditions and using (6), (11), and (12) to solve out for W_t / P_t, h_t, and D_t, section 1 of the appendix uses (1)-(5), (7), (9), (10), and (13)-(18) to form a system of 14 equations in the 14 variables Y_t, C_t, $\ _t$, r_t, M_t / P_t, Q_t, x_t, $\ _t$, g_t, $\ _t$, a_t, Z_t, u_t, and $\ _t$. Some of the real variables in this system inherit unit roots from the random walk (3) in the technology shock. However, the variables $y_t = Y_t / Z_t$, $c_t = C_t / Z_t$, $m_t = (M_t / P_t) / Z_t$, $q_t = Q_t / Z_t$, $\ _t = Z_t \ _t$, and $z_t = Z_t / Z_{t-1}$ remain stationary and, in the absence of shocks, the economy converges to a steady-state growth path, along which all of the stationary variables are constant, with $y_t = y$, $c_t = c$, $\ _t = \ $, $r_t = r$, $m_t = m$, $q_t = q$, $x_t = x$, $\ _t = \ $, $g_t = g$, $\ _t = \ $, $a_t = 1$, $z_t = z$, $u_t = 1$, and $\ _t = \ $ for all $t = 0; 1; 2; \ldots$.

Equations (6) and (13), in particular, can be combined with (9) to obtain the steady-state relationship

$$\ln(m) = \ln(\ m\ ) \quad \ _r (r \quad 1);$$

where

$$\ _r = \ \frac{\ }{r} \quad \frac{=}{\ \ 1}$$

Section 1 of the appendix also shows that the system consisting of (1)-(5), (7), (9), (10), and (13)-(18) can be log-linearized around the steady-state to describe how the economy responds to shocks. Let $\hat{y}_t = \ln(y_t = y)$, $\hat{c}_t = \ln(c_t = c)$, $\hat{\ }_t = \ln(\ _t = \ )$, $\hat{r}_t = \ln(r_t = r)$, $\hat{m}_t = \ln(m_t = m)$, $\hat{q}_t = \ln(q_t = q)$, $\hat{x}_t = \ln(x_t = x)$, $\hat{\ }_t = \ln(\ _t = \ )$, $\hat{g}_t = \ln(g_t = g)$, $\hat{\ }_t = \ln(\ _t = \ )$, $\hat{a}_t = \ln(a_t)$, $\hat{z}_t = \ln(z_t = z)$, $\hat{u}_t = \ln(u_t)$, and $\hat{\ }_t = \ln(\ _t = \ )$ denote the percentage deviation .op9s-6enachf 28.1

.9701 Tf 6.828 -1.79608

for all $t = 0; 1; 2; \ldots$.

Equations (19)-(22), which are log-linearized versions of (5), (7), (14), and (15), de ne the model's New Keynesian IS relationship linking movements in the output gap $\hat{x}_t$ to the real interest rate $\hat{r}_t \quad E_t \hat{}_{t+1}$, with backward-looking elements introduced through habit formation in the representative household's utility function. In the special case where $= 0$, so that habit formation is absent, these equations combine to yield the simpler, purely-forward looking speci cation

$$\hat{x}_t = E_t \hat{x}_{t+1} \quad (\hat{r}_t \quad E_t \hat{}_{t+1}) + (1 \quad {}_a)\hat{a}_t :$$

Meanwhile (23), the linearized form of (13), is the New Keynesian Phillips curve, again with a backward-looking component entering when > 0, so that sticky individual goods prices are indexed to past in ation. In (23), the cost push shock has been renormalized as $\hat{e}_t = (1= {}_p)\hat{}_t$ and the new parameterF32l.e022 component of income, captured by Z_t, more than the transitory component $\hat{}_t$. Once again, $_r$ is the interest semi-elasticity of money demand and d_t acts like a money demand shock. Finally, in this linearized system, (26) and (27) follow from (17) and (18) to determine the growth rate of the nominal money stock and aggregate output, and (28)-(31), which restate (2)-(4) and (10), govern the dynamics of the preference, productivity, money demand, and cost p[redacted] Federal Reserve held the federal funds rate in a range close to zero, the Taylor rule (24) is replaced in the estimated model by the zero interest rate condition

$$\hat{r}_t = \quad \ln(r): \tag{32}$$

Simila[redacted] monetary policy is described by a rule[redacted]

$$\hat{}_t = \ _{mm}\hat{}_{t\ 1} + \ _{m}\ \hat{}_{t\ 1} + \ _{mx}\hat{x}_{t\ 1}: \tag{33}$$

When $_{mm} = \ _{m} = \ _{mx} = 0$, (33) reduces to the same constant money growth rule studied

funds rate is dropped from the list of observable variables. For this interval, the model's solution depends not only on the structural parameters that enter into the New Keynesian model, but also on the duration, denoted by $_t$

deviation listed in table 1. In particular, prior distributions for and are centered at 0.5, with standard deviations large enough to allow for values closer to zero or one. The prior distributions for

4 Results

4.1 Bayesian Estimates

Table 2 summarizes the posterior distributions of the New Keynesian model's 16 structural parameters, while gure 1 displays more fully the posterior densities using blue bars, comparing them to the priors, described above and outlined in red. These posterior distributions assign more weight to higher values for the habit formation parameter and lower values for the price indexation parameter , compared to the priors. The posterior density for implies a much atter Phillips curve than does the prior, perhaps re ecting the muted response of in ation to more dramatic movements in real variables during and since the Great Recession.[7] At rst glance, the estimated money demand semi-elasticity appears quite large. However, with interest rates measured here in quarterly terms, $_r$ has to be divided by 4 to obtain the semi-elasticity with respect to the interest rate quoted, more conventionally, in annual terms. Thus, in fact, the posterior median of $_r = 13.4$ is quite similar to the semi-elasticity estimates, ranging from 3.17 to 3.66, obtained by Belongia and Ireland (2019a) from cointegrating money demand relationships for Divisia M2. Estimates of centered near 12 point to the importance of adjustment costs for real balances, con rming conclusions from Nelson (2002) and Andres, Lopez-Salido, and Nelson (2004, 2009).

Posterior estimates of the parameters $_r$, and $_x$ from the Taylor rule (24) imply an even larger degree of interest rate smoothing and a more balanced response of policy to changes in the output gap and in ation than suggested by the prior. Estimates of $_a$ and $_a$ suggest that non-monetary aggregate demand disturbances have been large and persistent over the sample period. Estimates of $_u$ and $_u$, meanwhile, show that even

[7]The formula displayed by Del Negro, Giannoni, and Schorfheide (2015, p.174) can be used together with information displayed in table A-2 of the appendix to that same paper to compute the Phillips curve slope coe cient (labeled) implied by the posterior mode from estimating both Smets and Wouters' (2007) medium-scale New Keynesian DSGE model and an extended version featuring additional nancial frictions developed speci cally to explain the behavior of in ation over the post-crisis period. The posterior mode at $= 0.0169$ found here is comparable to the modal value of $= 0.0120$ from the Smets-Wouters model but substantially larger than the modal value of $= 0.0018$ from the extended model with nancial frictions.

more highly persistent money demand shocks have been important, too. Earlier results from Ireland (2000), Collard and Dellas (2005), and Gal (2015) strongly suggest that these money demand shocks will become an important source of additional macroeconomic volatility when the estimated Taylor rule is replaced by one calling for a constant rate of money growth. Less certain, however, is whether a money growth rule of the more general form (33) can cope more successfully with these disturbances. Finally, in figure 2, the posterior density for σ_z, measuring the volatility of productivity shocks, tightens but remains centered near its prior mean, while the volatility parameters σ_e and σ_r for the cost push and monetary policy shocks appear smaller, relative to values initially suggested by the prior.

Figures 2 and 3 show that the posterior distributions for the expected durations of the zero nominal interest rate episode overlap heavily with the corresponding priors, reflecting the absence of the additional term structure data that Kulish, Morley, and Robinson (2017) use to sharpen their estimates of these parameters. While the macroeconomic data do contribute modestly to determining the shape of these posterior distributions, to a large extent the expected durations here are essentially calibrated based on the survey data used to formulate the priors. Even by themselves, however, these survey data are useful in incorporating into the estimated model the shift in expectations towards much longer durations of the zero nominal interest rate episode that Swanson and Williams (2014) observe in late 2010, as well as the gradual reduction in expected durations as the economy continued to recover in 2014 and 2015.

Figure 4 plots the median paths from the posterior distributions of the New Keynesian model's five structural disturbances.[8] Not surprisingly, the estimated model attributes the Great Recession, with its accompanying declines in inflation and interest rates, to a series of large, adverse preference shocks. Unfavorable productivity shocks also appear throughout the post-2008 period, contributing to weakness in real GDP growth but also explaining why

[8] These paths are constructed from draws from the posterior distribution for each shock, taken using Durbin and Koopman's (2002) simulation-smoother for the unobservable states, as described in part 6 of the appendix.

in ation did not fall even further.

Since the previous results presented by Ireland (2000), Collard and Dellas (2005), and Gal (2015) suggest that money demand shocks pose the biggest challenge to the success of monetary policies that focus on targeting money growth instead of interest rates, the middle row of gure 4 plots the median paths for both the money demand shock ^

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form in (33). Consistent with the estimates of $_u$ reported earlier, gure 4 con rms that these innovations have been large, frequently exceeding 2 percent in both directions, positive and negative. But while, for the sake of consistency, all of the model's estimated innovations are interpreted as unpredictable in the counterfactual scenarios discussed below, it should be noted that at least some of the apparent high-frequency volatility in money demand that shows up in the estimated time path for $"_{ut}$ in gure 4 re ects institutional changes that, to the fhete aatnc. aMos fheaositive

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all found that a constant money growth rule produced excess volatility after money demand shocks relative to an interest rate rule, none of them considered the alternative of a money growth rule that adapted exibly to changing macroeconomic conditions in the same manner as the Taylor rule. Relative to the Taylor rule, in fact, one potential advantage to more ex-ible money growth rules of the form shown in (33) is that they do not require the aggressive response to in ation needed by interest rate rules to ensure the stability of a unique rational expectations equilibrium. Instead, money growth rules can stabilize long-run in ation simply by pinning down the average rate of money growth and focusing more directly on stabilizing the output gap over shorter time horizons.

Though no exhaustive attempt has been made here to identify the optimal money growth rule, search over a grid of values for the parameters reveals that setting $_{mm}$ = 1, $_{m}$ = 0, and $_{mx}$ = 0:125 delivers impressive performance in response to the array of shocks estimated to have hit the US economy over the 1983:1-2018:3 sample period, while minimizing the duration and importance of the episode, during and following the nancial crisis and Great Recession, over which the short-term nominal interest rate uctuates in a range near zero. This rule, which specializes (33) as

$$\hat{\ }_t = \hat{\ }_{t\ 1} \quad 0{:}125\hat{x}_{t\ 1}; \qquad (34)$$

generates modest but highly persistent adjustments in money growth. These adjustments work, directly, to stabilize the output gap and, indirectly, to stabilize in ation as well.

The middle columns of table 3 summarize the posterior distributions of output growth, in ation, the nominal interest rate, the money growth rate, and the output gap after the estimated Taylor rule (24) is replaced by the exible money growth rule (34), holding all other parameters and disturbances xed at their estimated values. Thus, these counterfactual simulations confront the central bank with the same patterns of preference, productivity, money demand, and cost push shocks estimated to have hit the US economy over the 1983:1-

2018:3 sample period, but replace the Federal Reserve's historical policy of interest rate management, including the forward guidance used to lengthen the expected duration of the zero nominal interest rate episode, with the policy dictated by the flexible money growth rule instead.

As noted above, the form of the model's money demand relationship, implied by (9) and (25), allows the nominal interest rate to fall below zero in a well-defined rational expectations equilibrium. If the counterfactual path for the interest rate were to fall far below zero for an extended period of time, a concern might arise that the private financial system would adapt to profit from the spread between the zero interest rate on currency and the negative nominal interest rate on bonds. It will be confirmed below, however, that in each of the counterfactual scenarios considered here, the episode of negative nominal interest rates is moderate and, in fact, considerably shorter than the seven-year period during which the rate rule stabilizes inflation following a productivity shock; to do so, it produces the increase in money growth that Ireland (1996) shows is necessary to generate, under sticky prices, the efficient increase in output that keeps the output gap unchanged. Likewise, the flexible money growth rule (34) calls for a monetary expansion after a favorable productivity shock that allows output to adjust more efficiently and minimizes the response in inflation.

Figure 7 confirms that here, as in Poole's (1970) classic Keynesian analysis, the estimated interest rate rule, by holding the short-term nominal rate fixed, insulates output growth, inflation, and the output gap by fully accommodating a shock to money demand. The flexible money growth rule falls a bit short of achieving this ideal, but nevertheless generates a persistent increase in money supply growth that largely accommodates the increase in money demand. It is noteworthy that these stabilizing effects appear even though, under the flexible money growth rule, the central bank responds to the output gap with a one-quarter lag. To the extent that the central bank could detect money demand shocks within the quarter and respond to them directly, the rule's performance could be improved still further. Finally, figure 8 shows impulse responses to cost push shocks under (34) that come close to replicating those that appear under the estimated interest rate rule.

4.3 Constant Money Growth

Consistent with the earlier results from Ireland (2000), Collard and Dellas (2005), and Gal (2015), the results in last three columns of table 3 suggest strongly that macroeconomic volatility would have been amplified greatly if the Federal Reserve had followed a policy directed at holding the growth rate of Divisia M2 perfectly fixed by setting ${}_{mm} = {}_{m} = {}_{mx} = 0$ in (33), again holding all other parameters and disturbances fixed at their estimated values. Median estimates of the standard deviations of output growth and inflation under the constant money growth rate rule are more than 50 percent larger than those under the estimated policy rule. Volatility in the output gap, meanwhile, increases by a factor of three.

Figures 5-8 again add detail. In figure 5, the monetary tightening prescribed by both the estimated interest rate rule and the flexible money growth rule does not occur under the constant money growth rule. Hence, under constant money growth, output growth, inflation, and the output gap all display considerably more volatility in response to preference shocks.

0:80, similar the to target maintained by the Swiss National Bank over the entire period since 2015, in 2009:3.

The exible money growth rule (34), again by sharp contrast, delivers additional stimulus that would have closed the negative output gap by the end of 2009. The large money demand shock in 2011:3 temporarily pushes output back below its e cient level. As noted above, however, this estimated disturbance to money demand, though interpreted by the model as an exogenous and unpredictable shock, re ects legal and institutional developments known to policymakers in advance; it might have been anticipated and at least partially accommodated in actual practice. The exible rule still produces a smoother time path for money growth than that observed historically. Most importantly, like the constant money growth rule, it requires only four quarters of negative interest rates. Along the counterfactual path, the short-termit reun money2]TJ 0 -231.9552 Tf 72 708166.189[(80,)-3TJ/F17 11.9552 Tf 9.298 0 Td [(0)]T extent that changes in money growth do play a separate role in the monetary transmission mechanism, as suggested by the empirical results in Belongia and Ireland (2018a, 2018b), the case for money growth rules grows stronger. Second, the simulations in Belongia and Ireland (2018a) hold money growth constant during and after 2008, but at rates that are higher than full-sample historical average. Therefore, though they call for constant money growth over the post-2008 period, the policy rules considered previously share with the

alternative that, in the same spirit of the Taylor rule, adjusts the rate of money growth, modestly and gradually, in response to movements in the output gap. Even without a direct response to money demand shocks, this rule helps the central bank accommodate those disturbances and, more generally, allows monetary policy to pursue short-run stabilization objectives even as it maintains an environment of nominal stability through its choice of the long-run money growth rate.

Counterfactual simulations reveal that this exible money growth rule would have produced macroeconomic stability over the 1983:1-2018:3 sample period comparable to that observed, historically, under the estimated interest rate rule. Moreover, by targeting the rate of money growth and allowing interest rates to adjust, as needed, to maintain equilibrium in the market for bonds, the simulations show that this rule would have generated a more rapid recovery in both output and in ation after 2009, without resorting to forward guidance and with exceptionally low interest rates prevailing for only one year.

Notably, these bene cial e ects appear even in a standard New Keynesian model in which, by assumption, monetary policy actions are transmitted to the economy through their impact on interest rates and the stability of the money growth rate itself o ers no additional advantage. To the extent that other channels of monetary transmission, like those identi ed empirically by Belongia and Ireland (2018a, 2018b), operate in the US economy, policy rules focusing on money growth instead of interest rates may o er further advantages not captured here. And to the extent that the money demand disturbances interpreted as exogenous and unpredictable here re ect legal and institutional changes known in advance to the Fed, they could be accommodated even under a money growth0 -23.9054(an)28t for28(ericy)-368(2tabiill26(for)

Edge, Rochelle M. and Refet S. Gurkaynak. \How Useful Are Estimated DSGE Model Forecasts for Central Bankers?" *Brookings Papers on Economic Activity* (Fall 2010): 209-244.

Friedman, Milton. \The Role of Monetary Policy." *American Economic Review* 58 (March 1968): 1-17.

Gal, Jordi. *Monetary Policy, Inflation, and the Business Cycle*, 2nd Ed. Princeton: Princeton University Press, 2015.

Greenspan, Alan. \Remarks by Chairman Alan Greenspan: Rules vs. Discretionary Monetary Policy." Speech at the 15th Anniversary Conference of the Center for Economic Policy Research. Stanford: Stanford University, 5 September 1997.

Guerrieri, Luca and Matteo Iacoviello. \OccBin: A Toolkit for Solving Dynamic Models with Occasionally Binding Constraints Easily." *Journal of Monetary Economics* 70 (March 2015): 22-38.

Hallman, Jeffrey J., Richard D. Porter, and David H. Small. \Is the Price Level Tied to the M2 Monetary Aggregate in the Long-Run?" *American Economic Review* 81 (September 1991): 841-858.

Ireland, Peter N. \The Role of Countercyclical Monetary Policy." *Journal of Political Economy* 104 (August 1996): 704-723.

Ireland, Peter N. \Interest Rates, Inflation, and Federal Reserve Policy Since 1980." *Journal of Money, Credit, and Banking* 32 (August 2000, Part 1): 417-434.

Ireland, Peter N. \Technology Shocks in the New Keynesian Model." *Review of Economics and Statistics* 86 (November 2004): 923-936.

Ireland, Peter N. \Money's Role in the Monetary Business Cycle." *Journal of Money, Credit, and Banking* 36 (December 2004): 969-983.

Ireland, Peter N. \Changes in the Federal Reserve's Inflation Target: Causes and Consequences." *Journal of Money, Credit and Banking* 39 (December 2007): 1851-1882.

Ireland, Peter N. \On the Welfare Cost of Inflation and the Recent Behavior of Money Demand." *American Economic Review* 99 (June 2009): 1040-1052.

Ireland, Peter N. \A New Keynesian Perspective on the Great Recession." *Journal of Money, Credit and Banking* 43 (February 2011): 31-54.

Jackson, Harriet. \The International Experience with Negative Policy Rates." Staff Discussion Paper 2015-13. Ottawa: Bank of Canada, November 2015.

Judson, Ruth A., Bernd Schlusche, and Vivian Wong. \Demand for M2 at the Zero Lower Bound: The Recent U.S. Experience." Finance and Economics Discussion Series 2014-22. Washington: Federal Reserve Board, January 2014.

King, Robert G., Charles I. Plosser, and Sergio T. Rebelo. \Production, Growth and Business Cycles: I. The Basic Neoclassical Model." *Journal of Monetary Economics* 21 (March-May 1988): 195-232.

Kulish, Mariano, James Morley, and Tim Robinson. \Estimating DSGE Models with Zero Interest Rate Policy." *Journal of Monetary Economics* 88 (June 2017): 35-49.

Meulendyke, Ann-Marie. *U.S. Monetary Policy and Financial Markets*. New York: Federal Reserve Bank of New York, 1998.

Nelson, Edward. \Direct Effects of Base Money on Aggregate Demand: Theory and Evidence." *Journal of Monetary Economics* 49 (May 2002): 687-708.

Orphanides, Athanasios and Richard D. Porter. \P* Revisited: Money-Based Inflation Forecasts with a Changing Equilibrium Velocity." *Journal of Economics and Business* 52 (January-April 2000): 87-100.

Poole, William. \Optimal Choice of Monetary Policy Instruments in a Simple Stochasic Macro Model." [illegible] 84 (May 1970): 197-216.

Rognlie, Matthew. \What Lower Bound? Monetary Policy with Negative Interest Rates." Manuscript. Evanston: Northwestern University, July 2016.

Rotemberg, Julio J. \Sticky Prices in the United States." [illegible]

Table 1. Prior Distributions for Structural Parameters

Parameter		Distribution	Mean	Standard Deviation
Habit Formation		Beta	0.5	0.2
Price Indexation		Beta	0.5	0.2
Phillips Curve Slope		Gamma	0.1	0.03
Money Demand Semi-Elasticity	$_r$	Gamma	5	5
Money Demand Adjustment Cost		Gamma	10	10
Interest Rate Smoothing	$_r$	Beta	0.75	0.1
Policy Response to In ation		Gamma	0.4	0.1
Policy Response to Output Gap	$_x$	Gamma	0.2	0.1
Preference Shock Persistence	$_a$	Beta	0.75	0.1
Money Demand Shock Persistence	$_u$	Beta	0.75	0.1
Cost Push Shock Persistence	$_e$	Beta	0.5	0.1
Preference Shock Volatility	$_a$	Inverse Chi-squared	0.0125	0.0066
Productivity Shock Volatility	$_z$	Inverse Chi-squared	0.0125	0.0066
Money Demand Shock Volatility	$_u$	Inverse Chi-squared	0.0125	0.0066
Cost Push Shock Volatility	$_e$	Inverse Chi-squared	0.0031	0.0016
Monetary Policy Shock Volatility	$_r$	Inverse Chi-squared	0.0031	0.0016

Note: Prior distributions for the standard deviations $_i$, $i = a; z; u; e; r$, are those induced by assuming that the associated variance ^2_i has the inverse chi-squared distribution with scale parameter 0.01^2 for $i = a; z; u$ or 0.0025^2 for $i = e; r$ and 4 degrees of freedom.

Table 3. Counterfactual Simulations

	Estimated			Money Growth Rule			Constant Money Growth		
Standard Deviation of	Median	16	84	Median	16	84	Median	16	84
Output Growth	0.6032	0.6032	0.6032	0.6496	0.6259	0.6826	0.9194	0.8386	1.0253
Inflation	0.2451	0.2451	0.2451	0.2572	0.2336	0.2919	0.4132	0.3703	0.4600
Nominal Interest Rate	0.7131	0.7131	0.7131	0.6149	0.5667	0.6690	0.5006	0.4452	0.5716
Money Growth Rate	0.7479	0.7479	0.7479	0.5489	0.5156	0.5841	0.0000	0.0000	0.0000
Output Gap	0.7311	0.5716	0.9192	0.7967	0.7056	0.8989	2.4659	2.0456	2.9700

Note: The table shows the median and the 16th and 84th percentiles of the posterior distribution for the historical standard deviation of the indicated variable under the estimated policy rule, the flexible money growth rule (34) described in the text, and constant money growth

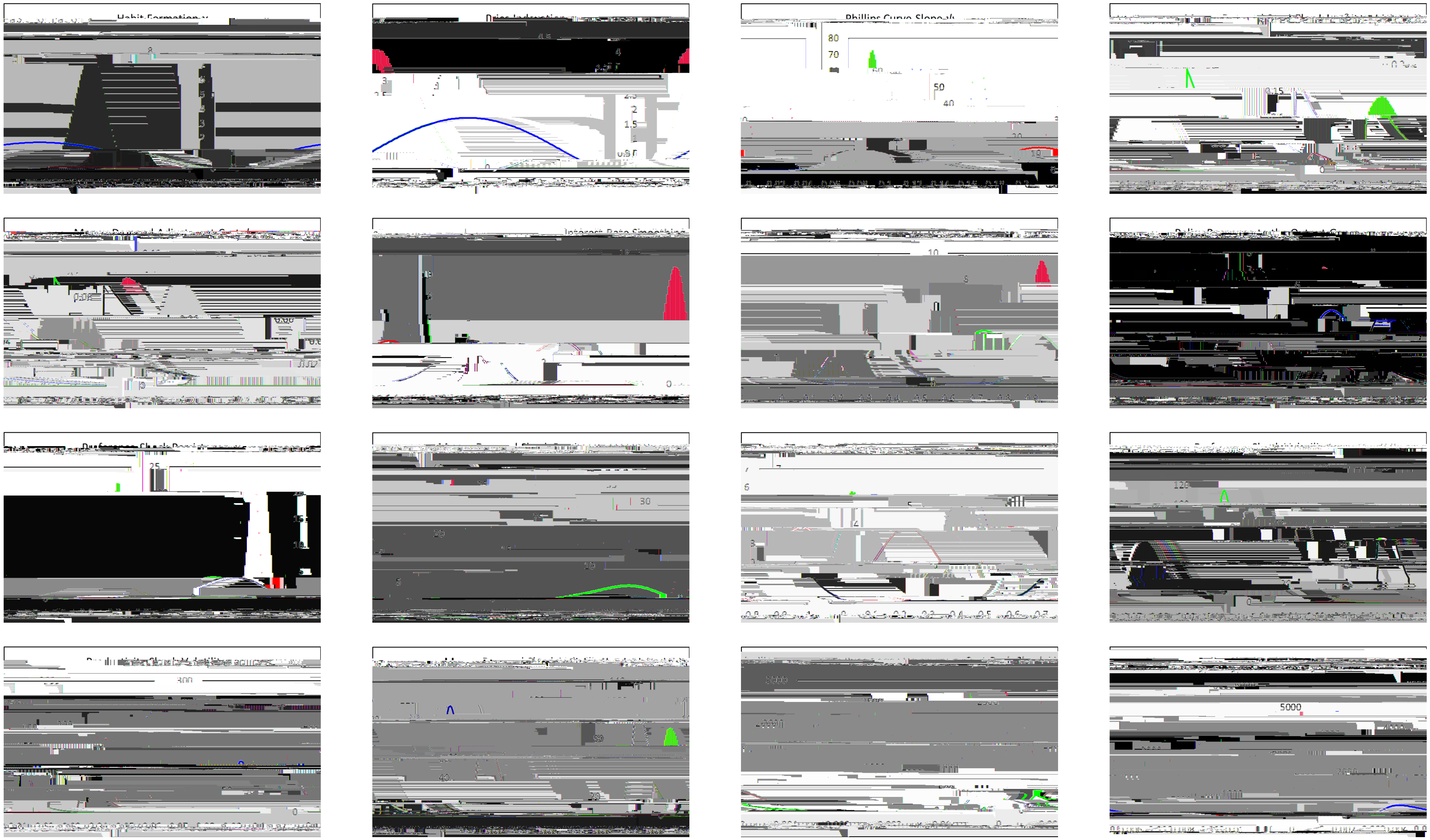

Figure 1. Prior and Posterior Densities, Structural Parameters. Each panel shows the prior (red line) and posterior (blue bars) density of the indicated structural parameter.

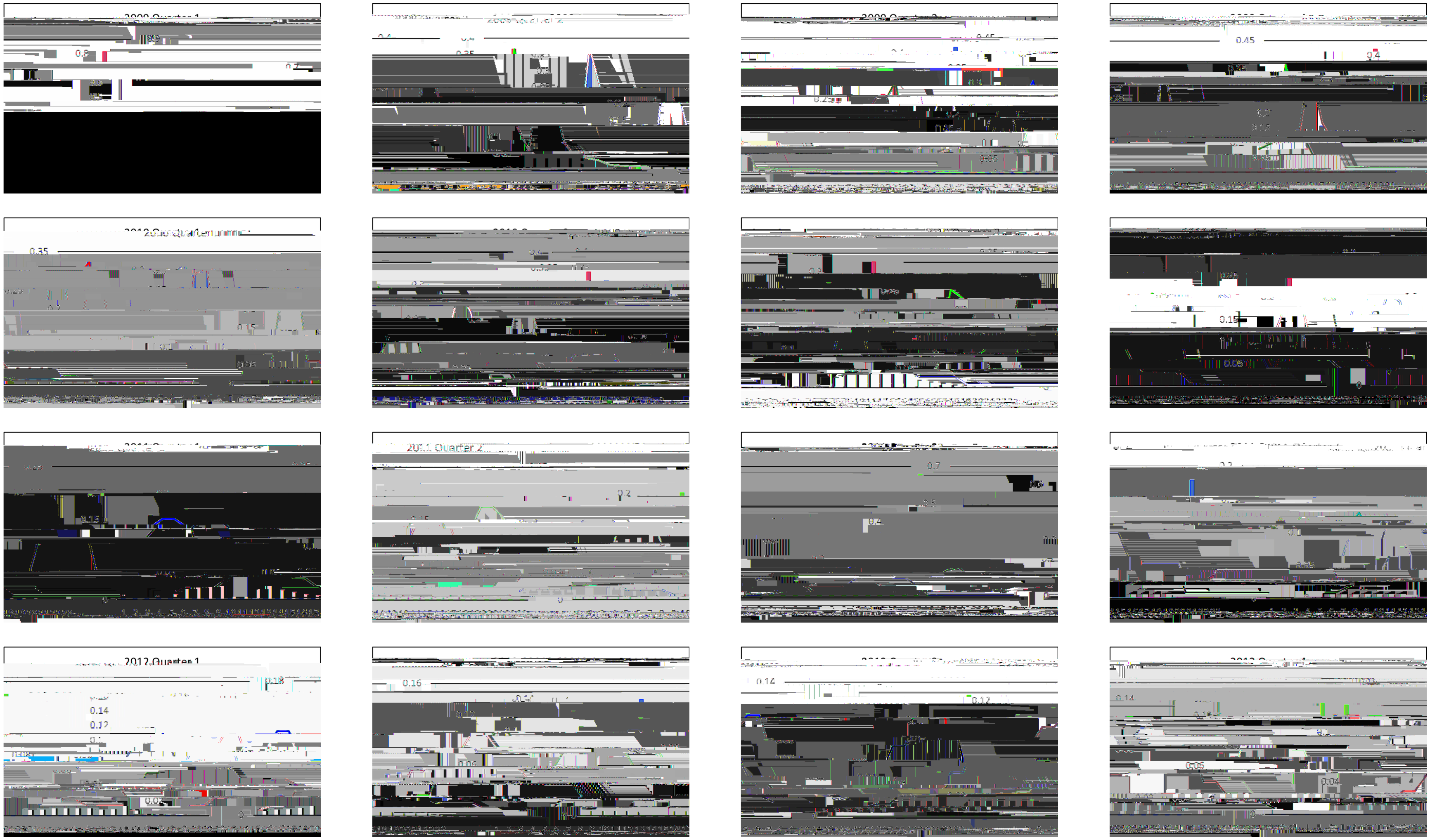

Figure 2. Prior and Posterior Densities, Expected Zero Nominal Interest Rate Episode. Each panel shows the prior (red line) and posterior (blue bars) density of the expected duration of the zero nominal interest rate episode at the indicated date.

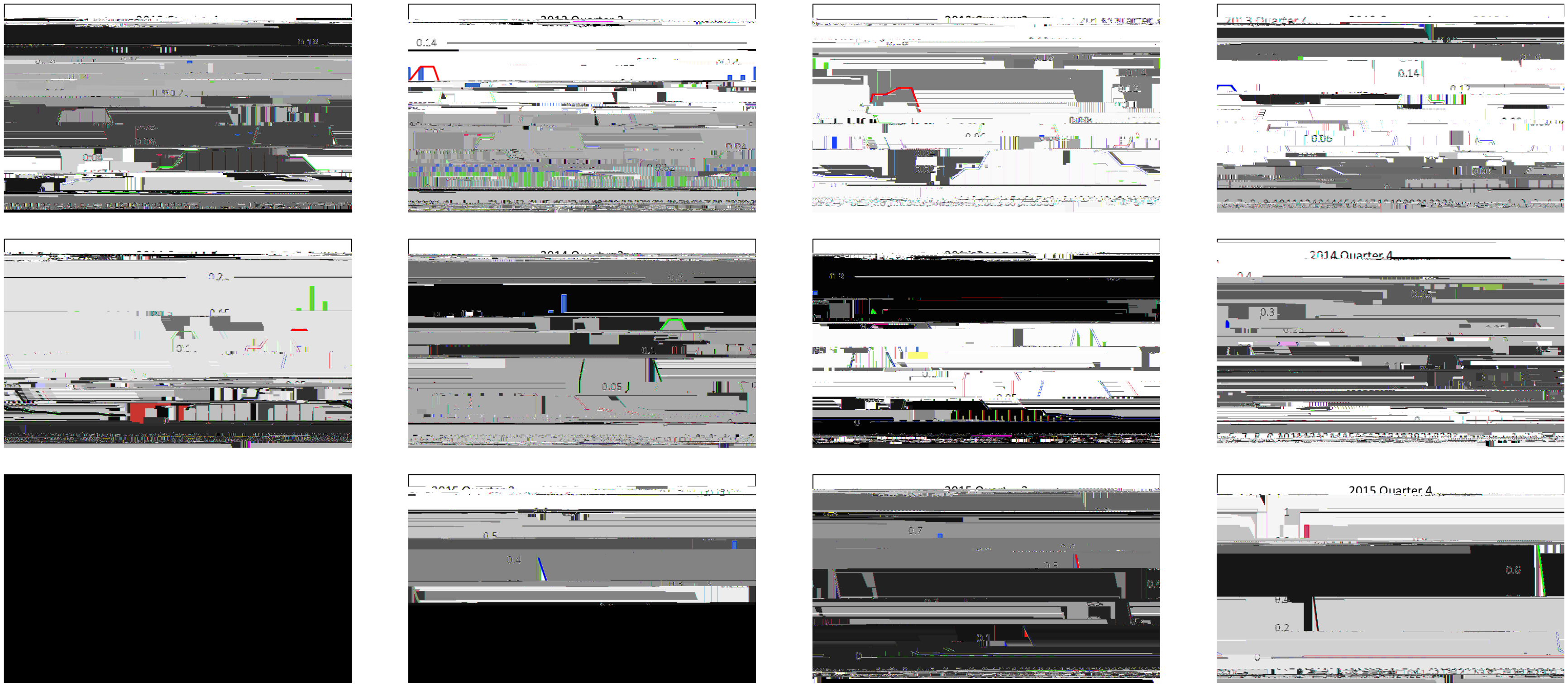

Figure 3. Prior and Posterior Densities, Expected Zero Nominal Interest Rate Episode. Each panel shows the prior (red line) and posterior (blue bars) density of the expected duration of the zero nominal interest rate episode at the indicated date.

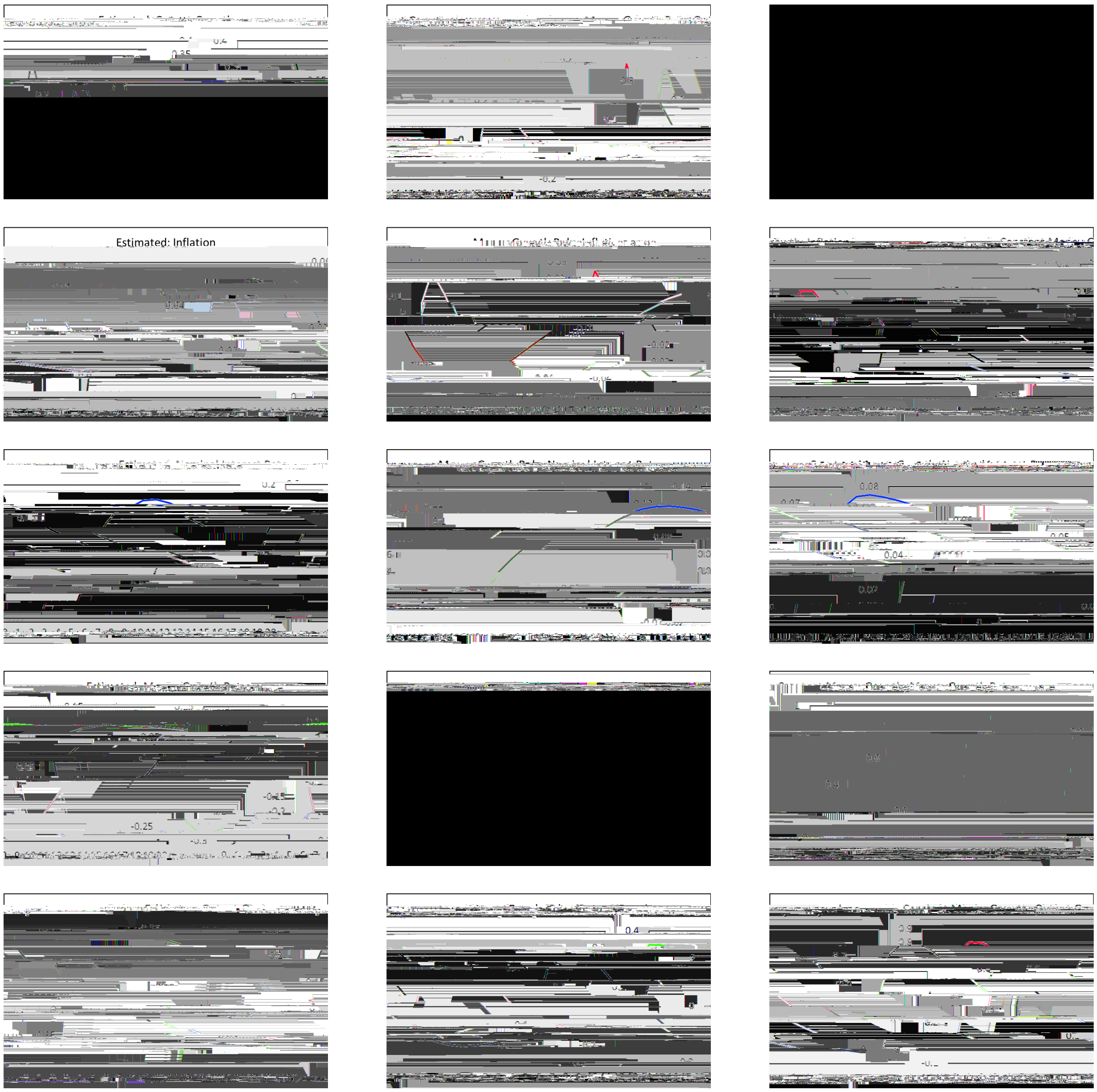

Figure 5. Impulse Responses to a Preference Shock. Each panel shows the percentage-point response of the indicated variable to a one-standard-deviation preference shock under the estimated policy rule, the exible money growth rule (34) described in the text, and constant money growth, when the parameters of the structural model are set equal to their posterior modes.

Figure 6. Impulse Responses to a Productivity Shock. Each panel shows the percentage-point response of the indicated variable to a one-standard-deviation productivity shock under the estimated policy rule, the exible money growth rule (34) described in the text, and constant money growth, when the parameters of the structural model are set equal to their posterior modes.

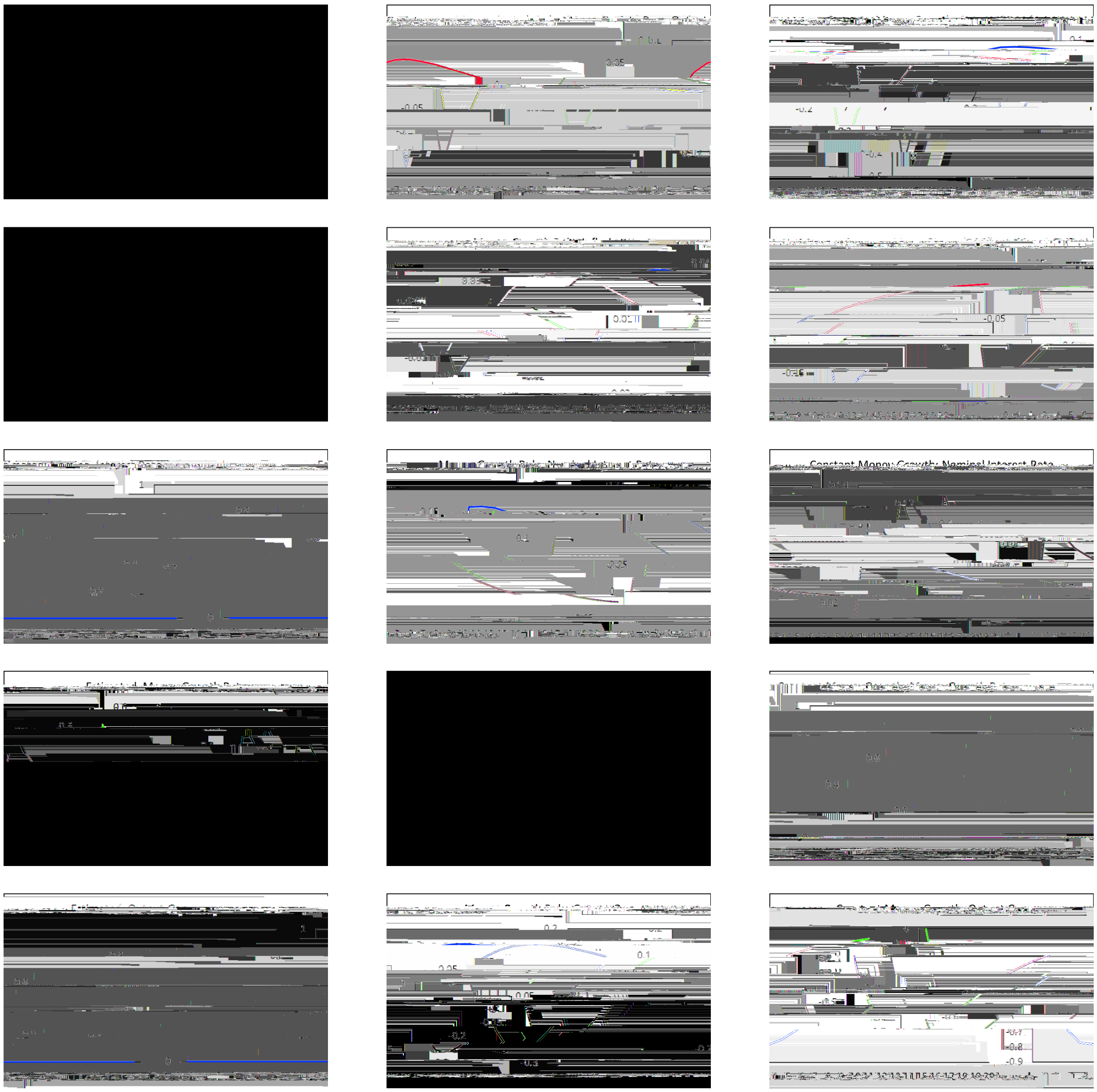

Figure 7. Impulse Responses to a Money Demand Shock. Each panel shows the percentage-point response of the indicated variable to a one-standard-deviation productivity shock under the estimated policy rule, the exible money growth rule (34) described in the text, and constant money growth, when the parameters of the structural model are set equal to their posterior modes.

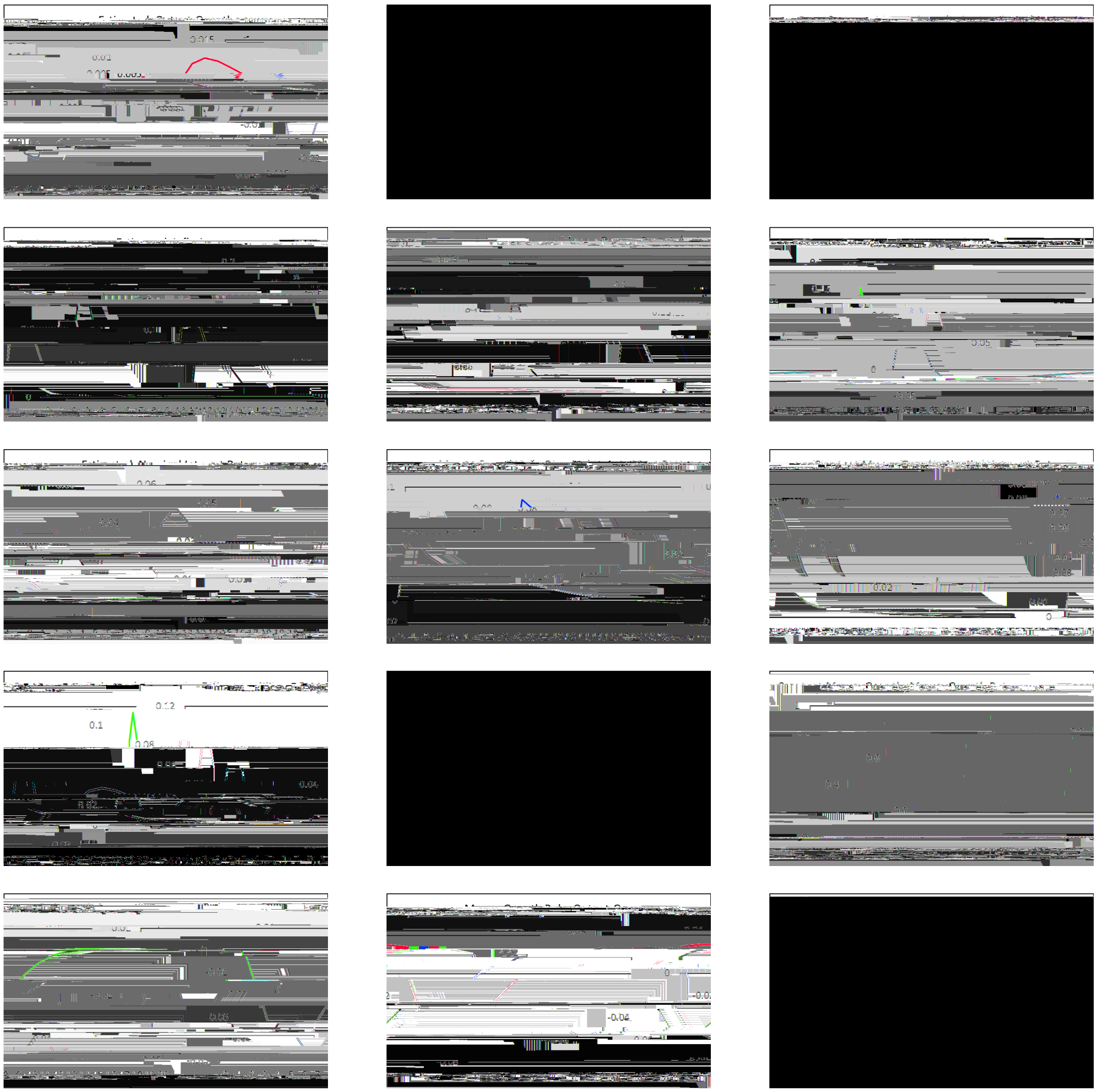

Figure 8. Impulse Responses to a Cost Push Shock. Each panel shows the percentage-point response of the indicated variable to a one-standard-deviation cost push shock under the estimated policy rule, the exible money growth rule (34) described in the text, and constant money growth, when the parameters of the structural model are set equal to their posterior modes.

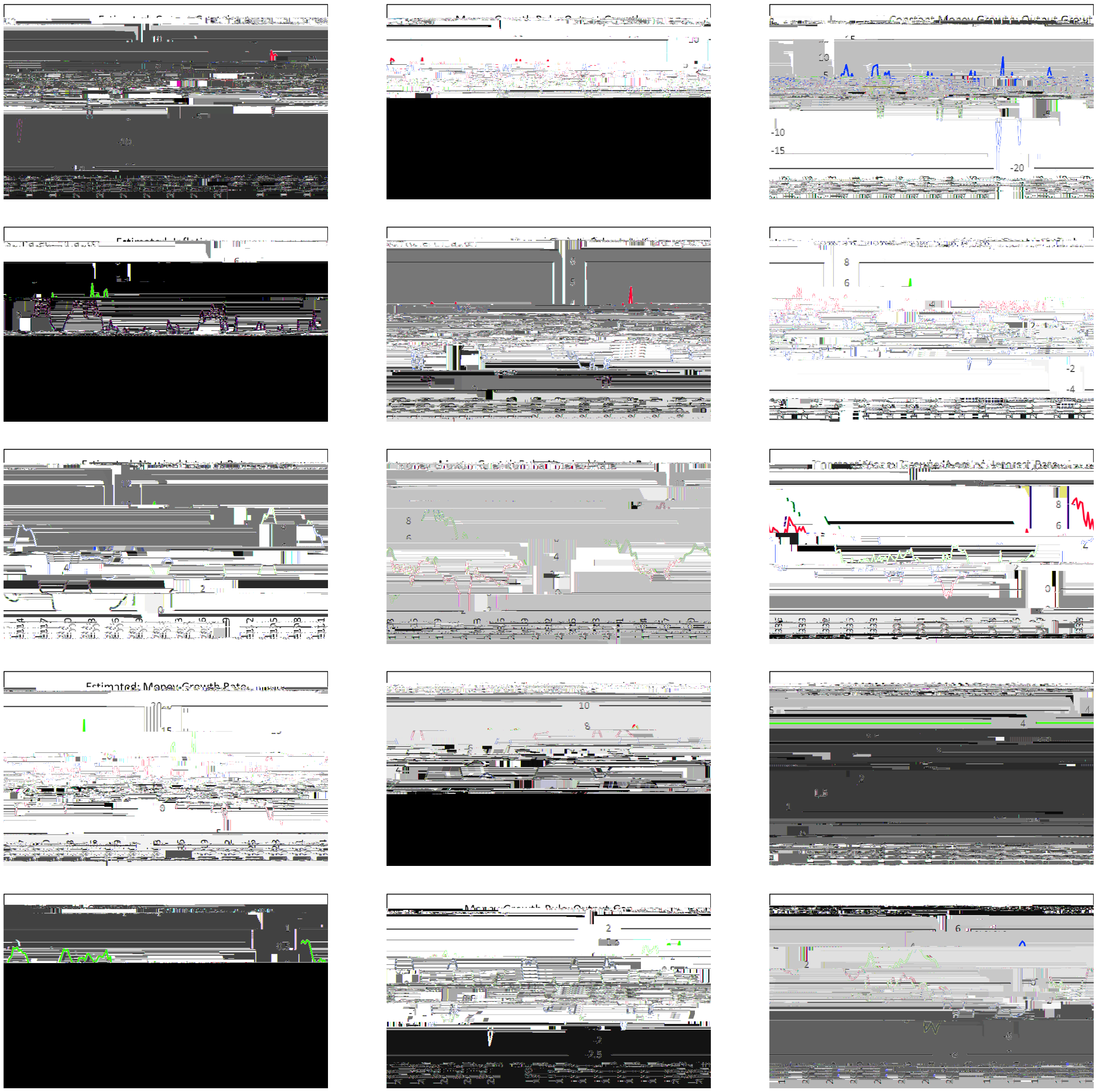

Figure 9. Counterfactual Simulations. Panels in the first column show the actual path for output growth, inflation, the nominal interest rate, and the money growth rate, all in annualized terms, and the median path from the estimated posterior distribution of the output gap. Panels in the second and third columns show median counterfactual paths from the estimated posterior distribution of the same variables under the flexible money growth rule (34) described in the text and constant money growth.

7 Appendix

7.1 Deriving the Log-Linearized Model

After imposing the symmetry and market clearing conditions $Y_t(i) = Y_t$, $h_t(i) = h_t$, $D_t(i) = D_t$, and $P_t(i) = P_t$ for all $i \in [0,1]$ and $t = 0,1,2,\ldots$ and $M_t = M_{t-1} + T_t$ and $B_t = B_{t-1} = 0$ for all $t = 0,1,2,\ldots$, (6), (11), and (12) can be used to solve out for W_t/P_t, h_t, and D_t. The system implied by (1)-(5), (7), (9), (10), and (13)-(18) then becomes

$$Y_t = C_t + \frac{\phi_p}{2}\left(\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}} - 1\right)^2 Y_t; \tag{1}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}; \tag{2}$$

$$\ln(Z_t) = \ln(z) + \ln(Z_{t-1}) + \varepsilon_{zt}; \tag{3}$$

$$\ln(u_t) = \rho_u \ln(u_{t-1}) + \varepsilon_{ut}; \tag{4}$$

$$\Lambda_t = \frac{a_t}{C_t - \gamma C_{t-1}} - \beta\gamma E_t\left(\frac{a_{t+1}}{C_{t+1} - \gamma C_t}\right); \tag{5}$$

$$\Lambda_t = \beta r_t E_t(\Lambda_{t+1}/\pi_{t+1}); \tag{7}$$

$$\begin{aligned}
&\frac{a_t}{\omega}\left[\ln(m) - \ln\left(\frac{M_t}{P_t Z_t}\right) + \ln(u_t)\right] - a_t\frac{\phi_m}{2}\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)^2 \\
&- a_t\phi_m\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}} - 1\right)\left(\frac{M_t/P_t}{zM_{t-1}/P_{t-1}}\right) \\
&+ \beta\phi_m E_t\left[a_{t+1}\left(\frac{M_{t+1}/P_{t+1}}{zM_t/P_t} - 1\right)\left(\frac{M_{t+1}/P_{t+1}}{zM_t/P_t}\right)^2\left(\frac{zZ_t}{Z_{t+1}}\right)\right] \\
&= Z_t\Lambda_t\left(1 - \frac{1}{r_t}\right);
\end{aligned} \tag{9}$$

$$\ln(\theta_t) = (1-\rho_\theta)\ln(\theta) + \rho_\theta\ln(\theta_{t-1}) + \varepsilon_{\theta t}; \tag{10}$$

$$\begin{aligned}
\theta_t - 1 = {} & \theta_t\left(\frac{a_t}{\Lambda_t Z_t}\right) - \phi_p\left(\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}} - 1\right)\left(\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}}\right) \\
& + \beta\phi_p E_t\left[\left(\frac{\Lambda_{t+1}Y_{t+1}}{\Lambda_t Y_t}\right)\left(\frac{\pi_{t+1}}{\pi_t^{\alpha}\pi^{1-\alpha}} - 1\right)\left(\frac{\pi_{t+1}}{\pi_t^{\alpha}\pi^{1-\alpha}}\right)\right];
\end{aligned} \tag{13}$$

$$\frac{1}{Z_t} = \frac{1}{Q_t - \gamma Q_{t-1}} - \beta\gamma E_t\left[\left(\frac{a_{t+1}}{a_t}\right)\left(\frac{1}{Q_{t+1} - \gamma Q_t}\right)\right]; \tag{14}$$

$$x_t = Y_t/Q_t; \tag{15}$$

$$\ln(r_t/r) = \rho_r\ln(r_{t-1}/r) + \rho_\pi\ln(\pi_{t-1}/\pi) + \rho_x\ln(x_{t-1}/x) + \varepsilon_{rt}; \tag{16}$$

$$\mu_t = \frac{M_t/P_t}{M_{t-1}/P_{t-1}}\pi_t; \tag{17}$$

and

$$g_t = Y_t/Y_{t-1} \tag{18}$$

for all $t = 0, 1, 2, \ldots$.

In terms of the stationary variables $y_t = Y_t/Z_t$, $c_t = C_t/Z_t$, $\ _t$, r_t, $m_t = (M_t/P_t)/Z_t$, $q_t = Q_t/Z_t$, x_t, $\ _t$, g_t, $\ _t = Z_t\ _t$, a_t, $z_t = Z_t/Z_{t-1}$, u_t, and $\ _t$, the system of symmetric equilibrium conditions can be rewritten as

$$y_t = c_t + \frac{p}{2}\left(\frac{t}{t-1} - 1\right)^2 y_t; \tag{1}$$

$$\ln(a_t) = \ _a \ln(a_{t-1}) + \varepsilon_{at}; \tag{2}$$

$$\ln(z_t) = \ln(\ z) + \varepsilon_{zt}; \tag{3}$$

$$\ln(u_t) = \ _u \ln(u_{t-1}) + \varepsilon_{ut}; \tag{4}$$

$$\ _t = \frac{a_t z_t}{z_t c_t - \ c_{t-1}} - \ E_t \left(\frac{a_{t+1}}{z_{t+1} c_{t+1} - \ c_t}\right); \tag{5}$$

$$\ _t = \ r_t \ _t$$

for all $t = 0; 1; 2; \ldots$.

The stationary system pins down the steady-state values $y_t = y$, $c_t = c$, t

simpler form

$$s_{0;t} = As_{0;t\;\;1} + BE_t s_{0;t+1} + C\;\;_t; \qquad \text{(A.2)}$$

where $A = A_0^{\;1} A_1$, $B = A_0^{\;1} B_0$, and $C = A_0^{\;1}$

Finally, combining (A.4) and (A.9) yields

$$s_{t+1} = \quad s_t + W\varepsilon_{t+1}; \qquad \text{(A.10)}$$

where

$$s_t = \begin{bmatrix} s_{0,t}' & \quad_t' \end{bmatrix}' = \begin{bmatrix} \hat{y}_t & \hat{\ }_t & \hat{r}_t & \hat{m}_t & \hat{g}_t & \hat{\ }_t & \hat{q}_t & \hat{x}_t & \hat{\ }_t & \hat{a}_t & \hat{z}_t & \hat{u}_t & \hat{e}_t & \varepsilon_{rt} \end{bmatrix}';$$

$$= \begin{bmatrix} D & HP \\ 0_{(5\times 9)} & P \end{bmatrix};$$

and

$$W = \begin{bmatrix} H \\ I_{(5\times 5)} \end{bmatrix}.$$

It only remains to find the matrix D that solves (A.6). To accomplish this task, start by rewriting (A.2) as

$$KE_t s_{1,t+1} = \quad I_{(5}$$

solution; and if less than nine of the generalized eigenvalues lie outside the unit circle, then the system has multiple stable solutions. For details, see Blanchard and Kahn (1980) and Klein (2000).

Assuming that there are exactly nine generalized eigenvalues that lie outside the unit circle, partition the matrix Z into 9 9 blocks:

$$Z = \begin{matrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{matrix} :$$

Then, according to Higham and Kim (2000) and Lan and Meyer-Godhe (2012),

$$D = Z_{21} Z_{11}^{1} \quad (A.12)$$

will be the unique solution to (A.6) with all of its eigenvalues inside the unit circle, and the matrix F appearing in (A.7) will also have all of its eigenvalues inside the unit circle.

7.3 Imposing Zero Nominal Interest Rates

Kulish, Morley, and Robinson (2017) outline methods to solve and estimate the model over samples including the period from 2009:1 through 2015:4 when the Federal Reserve held short-term nominal interest rates in the US in a range near zero. Prior to and after the zero nominal interest rate period, the log-linearized model's solution is given by (A.10), as derived above. Lett = T_1 denote the start of the zero interest rate period, when the central bank replaces the Taylor rule (24) with the zero nominal interest rate condition (32). Then (32) can be combined with the remaining equilibrium conditions (19)-(23) and (24)-(31) to obtain

$$A_0 s_{0;t} = J_0 + A_1 s_{0;t \quad 1} + B_0 E_t s_{0;t+1} + C_0 \quad {}_t ; \quad (A.13)$$

where the 9Td [(C)]TJ/F30 7.9701 T Tf452 9 Td [(9)-324(thtrix)c

Substitute (A.16) into (A.14) to obtain

$$s_{0;t} = J + As_{0;t\ \ 1} + BJ_{t+1} + BD_{t+1}s_{0;t} + BH_{t+1}P\ \ _t + C\ \ _t: \qquad \text{(A.17)}$$

Matching coe cients across (A.15) and (A.16) then yields

$$D_t = [I_{(9\ \ 9)}\ \ \ BD_{t+1}]\ ^1A; \qquad \text{(A.18)}$$

$$H_t = [I_{(9\ \ 9)}\ \ \ BD_{t+1}]\ ^1(C + BH_{t+1}P); \qquad \text{(A.19)}$$

and

$$J_t = [I_{(9\ \ 9)}\ \ \ BD_{t+1}]\ ^1(J + BJ_{t+1}): \qquad \text{(A.20)}$$

Starting from the terminal conditions $D_{T_2+1} = D$ and $H_{T_2+1} = H$, where D is determined by (A.12) and H by (A.8), and $J_{T_2+1} = 0_{(9\ \ 1)}$, (A.18)-(A.20) can be solved via backward recursion for the sequences $f D_{T_1+j}\, g_{j=0}^{\ \ 1}$, $f H_{T_1+j}\, g_{j=0}^{\ \ 1}$, and $f J_{T_1+j}\, g_{j=0}^{\ \ 1}$, that appear in (A.15).

Still following Kulish, Morley, and Robinson (2017), assume more generally that the central bank re-evaluates the timing of its return to conventional policymaking via the Taylor rule (24) each period, announcing at the beginning of each time period t that the zero nominal interest rate episode will continue for $_t$ more periods. To keep track of outcomes in this case, let be an arbitrarily large upper bound on the length of the zero interest rate episode, and re-label the subscripts on the matrices that solve (A.18)-(A.20) to that $f D_k g_{k=1}$, $f H_k g_{k=1}$, and $f J_k g_{k=1}$ are those that apply during any period when the zero interest rate episode is expected to last for k more periods. Now, the matrices that appear in the solution (A.15) for the zero interest rate episode are given by $D_t = D\ _t$, $H_t = H\ _t$, and $J_t = J\ _t$. And, as noted

is the covariance matrix of the New Keynesian model's structural shocks.

The innovations $f \;{}_t g_{t=1}^{T}$ can then be used to evaluate likelihood function as

$$\ln(L(f d_t g_{t=1}^{T} j \;\; ; \;)) = \frac{4(T \quad T_2 + T_1 \quad 1) + 3(T_2 \quad T_1 + 1)}{2} \ln(2 \;) \quad \frac{1}{2}\sum_{t=1}^{T} \ln(j U_t \;\; {}_t U_t^0 j) \quad \frac{1}{2}\sum_{t=1}^{T} {}_t^0 (U_t \;\; {}_t U_t^0)^{\;1} \;{}_t :$$

7.5 Simulating the Posterior Distribution

The log posterior kernel can be evaluated as

$$\ln L(f d_t g_{t=1}^{T} j \;\; ; \;) + \ln(\;\; P(\;\; ; \;)) \;\; ;$$

where $P(\;\; ; \;)$ is the prior density over both sets of parameters. Kulish, Morley, and Robinson's (2017) modi cation of the randomized block Metropolis-Hastings algorithm of Chib and Ramamurthy (2010) is used to simulate draws from this posterior distribution. The algorithm treats and as separate blocks of parameters; this is natural, as consists of continuously-valued structural parameters where as the durations in are restricted to the positive integers.

The algorithm is initialized by nding the mode $\hat{}_0$ of the log posterior kernel, evaluated using data running from 1983:1 through 2008:4, that is, before the zero nominal interest rate episode, and , minus one times the inverse of the matrix of second derivatives of the log posterior kernel, evaluated at this initial maximizer. Similarly, the mode of the prior distributions for each of the duration parameters is used to initialize $\hat{}_0$.

A random number n of the 16 parameters in get updated in each iteration of the algorithm. First, n itself is chosen from a discrete uniform distribution over $[1; 16]$. Next, the speci c n parameters to be updated are randomly chosen without replacement, again from a discrete uniform distribution over $[1; 16]$. Using $^{(1)}$ to denote the vector of parameters to be updated and $^{(2)}$ the vector of parameters that are not being updated, and given the previous draw $\hat{}_i = (\hat{}_i^{(1)} i$

With $_{i+1}^{(1)}$ drawn from this conditional distribution and with

$$! = \min \left(\frac{L(f\, d_t g_{t=1}^{T}\, j\ _{i+1}^{(1)};\ _{i+1}^{(2)};\ \hat{}\ _i)P(\ _{i+1}^{(1)})}{L(f\, d_t g_{t=1}^{T}\, j\ \hat{}_i^{(1)};\ \hat{}_i^{(2)};\ \hat{}\ _i)P(\hat{}_i^{(1)})}; 1 \right) ;$$

' is drawn from a continuous uniform distribution on $(0;1)$. If $' > !$, the new draw is rejected by setting $\hat{}_{i+1} = \hat{}_i$. If '

Durbin and Koopman show that the sequence $\{\hat{\varepsilon}_t\}_{t=1}^{T+1}$ constructed using

$$\hat{\varepsilon}_{t+1} = \varepsilon^{a}_{t+1} - \tilde{\varepsilon}^{a}_{t+1} + \tilde{\varepsilon}_{t+1}$$

are draws from the posterior distribution of the vector $\{\varepsilon_t\}_{t=1}^{T+1}$ of innovations to the New Keynesian model's structural shocks, conditional on the entire series of observed data $\{d_t\}^{T}$

Higham, Nicholas J. and Hyun-Min Kim. \Numerical Analysis of a Quadratic Matrix Equation." IMA J of Num Anal