INTERDISTRICT SCHOOL CHOICE: A THEORY OF STUDENT ASSIGNMENT y

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Abstract. Interdistrict school choice programs|where a student can be assigned to a school outside of her district|are widespread in the US, yet the market-design literature has not considered such programs. We introduce a model of interdistrict school choice and present two mechanisms that produce stable or e cient assignments. We consider three categories of policy goals on assignments and identify when the mechanisms can achieve them. By introducing a novel framework of interdistrict school choice, we provide a new avenue of research in market design.

1. Introduction

School choice is a program that uses preferences of children and their parents over public schools to assign children to schools. It has expanded rapidly in the United States and many other countries in the last few decades. Growing popularity and interest in school choice stimulated research in market design, which has not only studied this problem in the abstract, but also contributed to designing speci c assignment mechanisms. [1]

Existing market-design research about school choice is, however, limited to intradistrict choice, where each student is assigned to a school only in her own district. In other words, the literature has not studied interdistrict choice, where a student can be assigned to a school outside of her district. This is a severe limitation for at least two reasons. First, interdistrict school choice is widespread: some form of it is practiced in 43 U.S. states. [2] Second, as we illustrate in detail below, many policy goals in school choice impose constraints across districts in reality, but the existing literature assumes away such constraints. This omission limits our ability to analyze these policies of interest.

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[1] See Abdulkadiroglu et al. (2005a,b, 2009) for details of the implementation of these new school choice procedures in New York and Boston.

[2] See http://ecs.force.com/mbdata/mbquest4e?rep=OE1705 , accessed on July 14, 2017.

Figure 1. Minnesota-Saint Paul metro area school districts participating in the AI program. The districts with the same color are adjoining districts that exchange students with one another.

rationality and strategy-proofness.[8] We first demonstrate an impossibility result; when the diversity policy is given as type-specific ceilings at the district level, there is no mechanism that satisfies the policy goal, constrained efficiency, individual rationality, and strategy-proofness. By contrast, a version of the top trading cycles (TTC) mechanism (Shapley and Scarf, 1974) satisfies these properties when the policy goal satisfies M-convexity, a concept in discrete mathematics (Murota, 2003). We proceed to show that the balanced-exchange policy and an alternative form of diversity policy—type-specific ceilings at the individual school level instead of at the district level—are M-convex, so TTC satisfies the desired properties for these policies. The same conclusion holds even when both of these policy goals are imposed simultaneously.

[8]Without individual rationality, all the other desired properties can be attained by a serial dictatorship.

We also consider the case when there is a policy function that measures how well a matching satisfies the policy goal. For example, diversity of a matching can be measured as its distance to an ideal distribution of students. We show that TTC satisfies the same desirable properties when the policy function satisfies *pseudo M-concavity*, a notion of concavity for discrete functions that we introduce. Furthermore, we show that there is an equivalence between two approaches based on the M-convexity of the policy set and the pseudo M-concavity of the policy function. Therefore, both results can naturally be applied in different settings depending on how the policy goals are stated.

Related Literature. Our paper is closely related to the controlled school choice literature that studies student diversity in schools in a given district. Abdulkadiroğlu and Sönmez (2003) introduce a policy that imposes type-specific ceilings on each school. This policy has been analyzed by Abdulkadiroğlu (2005), Ergin and Sönmez (2006), and Kojima (2012), among others. More accommodating policies using reserves rather than type-specific ceilings have been proposed and analyzed by Hafalir et al. (2013) and Ehlers et al. (2014). The latter paper finds difficulties associated with hard floor constraints, an issue further analyzed by Fragiadakis et al. (2015) and Fragiadakis and Troyan (2017).[9] In addition to sharing the motivation of achieving diversity, our paper is related to this literature in that we extend the type-specific reserve and ceiling constraints to district admissions rules. In contrast to this literature, however, our policy goals are imposed on districts rather than individual schools, which makes our model and analysis different from the existing ones.

The feature of our paper that imposes constraints on sets of schools (i.e., districts), rather than individual schools, is shared by several recent studies in matching with constraints. Kamada and Kojima (2015) study a model where the number of doctors who can be matched with hospitals in each region has an upper bound constraint. Variations and generalizations of this problem are studied by Goto et al. (2014, 2017), Biro et al. (2010), and Kamada and Kojima (2017, 2018), among others. While sharing the broad interest in constraints, these papers are different from ours in at least two major respects. First, they do not assume a set of hospitals is endowed with a well-defined choice function, while each school district has a choice function in our model. Second, the policy issues studied in these papers and those studied in ours are different given differences in the intended applications. These differences render our analysis distinct from those of the other papers, with none of their results implying ours and vice versa.

One of the notable features of our model is that district admissions rules do not necessarily satisfy the standard assumptions in the literature, such as substitutability, which

[9] In addition to the works discussed above, recent studies on controlled school choice and other two-sided matching problems with diversity concerns include Westkamp (2013), Echenique and Yenmez (2015), Sönmez (2013), Kominers and Sönmez (2016), Dur et al. (2014), Dur et al. (2016), and Nguyen and Vohra (2017).

guarantee the existence of a stable matching. In fact, even a seemingly reasonable district admissions rule may violate substitutability because a district can choose at most one contract associated with the same student—namely just one contract representing one school that the student can attend. Rather, we make weaker assumptions following the approach of Hatfield and Kominers (2014). This issue is playing an increasingly prominent role in matching with contracts literature; for example, in matching with constraints (Kamada and Kojima, 2015), college admissions (Aygün and Turhan, 2016; Yenmez, 2018), and post-graduate admissions (Hassidim et al., 2017), to name just a few.

Our analysis of Pareto efficient mechanisms is related to a small but rapidly growing literature that uses discrete optimization techniques for matching problems. Closest to ours is Suzuki et al. (2017), who show that a version of the TTC mechanism satisfies desirable properties if the constraint satisfies M-convexity. [10] Our analysis on efficiency builds upon and generalizes theirs. While the use of discrete convexity concepts for studying efficient object allocation is still rare, it has been utilized in an increasing number of matching problems such as two-sided matching with possibly bounded transfer (Fujishige and Tamura, 2006, 2007), matching with substitutable choice functions (Murota and Yokoi, 2015), matching with constraints (Kojima et al., 2018a), and trading networks (Candogan et al., 2016).

There is also a recent literature on segmented matching markets in a given district. Manjunath and Turhan (2016) study a setting where different clearinghouses can be coordinated, but not integrated, in a centralized clearinghouse and show how a stable matching can be achieved. In a similar setting, Dur and Kesten (2018) study sequential mechanisms and show that these mechanisms lack desired properties. In another work, Ekmekci and Yenmez (2014) study the incentives of a school to join a centralized clearinghouse. In contrast to these papers, we study which interdistrict school choice policies can be achieved when districts are integrated.

At a high level, the present paper is part of research in resource allocation under constraints. Real-life auction problems often feature constraints (Milgrom, 2009), and a great that we also analyze—while modeling exchanges of members of different institutions under constraints. Although the differences in the model primitives and exact constraints make it impossible to directly compare their studies with ours, these papers and ours clearly share broad interests in designing mechanisms under constraints.

The rest of the paper is organized as follows. Section 2 introduces the model. In Sections 3 and 4, we study when the policy goals can be satisfied together with stability and constrained efficiency, respectively. Section 5 concludes. Additional results, examples, and omitted proofs are presented in the Appendix.

2. Model

In this section, we introduce our concepts and notation.

2.1. Preliminary Definitions. There exist finite sets of students S, districts D, and schools C. Each student s and school c has a home district denoted by $d(s)$ and $d(c)$, respectively. Each student s has a type $\tau(s)$ that can represent different aspects of the student such as the gender, race, socioeconomic status, etc. The set of all types is finite and denoted by T. Each school c has a capacity q_c, which is the maximum number of students that the school can enroll. There exist at least two school districts with one or more schools. For each district d, k_d is the number of students whose home district is d. In each district, schools have sufficiently large capacities to accommodate all students from the district, i.e., for every district d, $k_d \leq \sum_{c:d(c)=d} q_c$. For each type t, k^t is the number of type-t students.

associated with s. Furthermore, we assume that the outside option is the least preferred outcome, so for every contract x associated with s, $x \; P_s \; \emptyset$. The corresponding weak order is denoted by R_s. More precisely, for any two contracts x, y associated with s, $x \; R_s \; y$ if $x \; P_s \; y$ or $x = y$.

A *matching* is a set of contracts. A matching X is *feasible for students* if there exists at most one contract associated with every student in X. A matching X is *feasible* if it is feasible for students and the number of contracts associated with every school in X is at most its capacity, i.e., for any $c \in C$, $|X_c| \leq q_c$. We assume that there exists a feasible *initial matching* $\tilde{X}$ such that every student has exactly one contract.[12] For any student s, if $\tilde{X}_s = \{(s, d, c)\}$ for some district d and school c, then c is called the *initial school* of s.

A *problem* is a tuple $(S, D, C, T, \{d(s), \tau(s), P_s\}_{s \in S}, \{Ch_d\}_{d \in D}, \{d(c), q_c\}_{c \in C}, \tilde{X})$. In what

$|Ch_d(X)| \geq |Ch_d(Y)|$.[14] A *completion* of a district admissions rule Ch_d is another admissions rule Ch'_d such that for every matching X either $Ch'_d(X)$ is equal to $Ch_d(X)$ or it is not feasible for students (Hatfield and Kominers, 2014). Throughout the paper, we assume that district admissions rules have completions that satisfy substitutability and LAD.[15] In Appendix B, we provide classes of district admissions rules that satisfy our assumptions.

2.3. Matching Properties, Policy Goals, and Mechanisms. A feasible matching X satisfies *individual rationality* if every student weakly prefers her outcome in X to her initial school, i.e., for every student s, $X_s \; R_s \; \tilde{X}_s$.

A *distribution* $\xi \in \mathbb{Z}_+^{|C||T|}$ is a vector such that the entry for school c and type t is denoted by ξ_c^t. The entry ξ_c^t is interpreted as the number of type-t students in school c at ξ. Furthermore, $\xi_d^t \equiv \sum_{c:d(c)=d} \xi_c^t$ denotes the number of type-t students in district d at ξ. Likewise, for any feasible matching X, the *distribution associated with* X is $\xi(X)$ whose $c;t$ entry $\xi_c^t(X)$ is the number of type-t students assigned to school c at X. Similarly, $\xi_d^t(X)$ denotes the number of type-t students assigned to district d at X.

We represent a distributional policy goal as a set of distributions. Let Ξ denotey mechanism is denoted as $_s(P_S)$. A mechanism satis es *strategy-proofness* if no student can misreport her preferences and get a strictly more preferred contract. More formally, for every student s and preference pro le P_S, there exists no preference P_s^0 such that $_s(P_s^0; P_{S n f s g})$ P_s $_s(P_S)$. For any property on matchings, a mechanism satis es the property if, for every preference pro le, the matching produced by the mechanism satis es the property.

3. Achieving Policy Goals with Stable Outcomes

To achieve stable matchings with desirable properties, we use a generalization of the deferred-acceptance mechanism of Gale and Shapley (1962).

Student-Proposing Deferred Acceptance Mechanism (SPDA).

Step 1: Each student s proposes a contract $(s; d; c)$ to district d where c is her most preferred school. Let X_d^1 denote the set of contracts proposed to district d. District d tentatively accepts contracts in $Ch_d(X_d^1)$ and permanently rejects the rest. If there are no rejections, then stop and return $[_{d2D} Ch_d(X_d^1)$ as the outcome.

Step n (n (

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Ccontract. s; d;

the school's capacity while ignoring the contracts of the students who have already been accepted at c_1. Likewise, district d_2 prioritizes students according to the order s_3 s_4 s_1 s_2 and chooses as many applicants as possible without going over the capacity of school c_3. These admissions rules are feasible and acceptant, and they have completions that satisfy substitutability and LAD. [16] In addition, student preferences are given by the following table,

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}
c_1	c_3	c_1	c_2
c_2	c_1	c_2	c_1
c_3	c_2	c_3	c_3

which means that, for instance, student s_1 prefers c_1 to c_2 to c_3.

In this problem, SPDA runs as follows. At the rst step, student s_1 proposes to district d_1 with contract $(s_1; c_1)$, student s_2 proposes to district d_2 with contract $(s_2; c_3)$, student s_3 proposes to district d_1 with contract $(s_3; c_1)$, and student s_4 proposes to district d_1

If individual rationality is violated so that some students prefer their initial schools to the outcome of SPDA, then there may be public opposition that harm interdistrict school choice e orts. For this reason, individual rationality is a desirable property for policymakers. The following condition proves to play a crucial role for achieving this property.

De nition 1. the Ih-26730(properen)-730nextnato always accepted. With this modi cation, it is easy to check that the outcome of SPDA is $\{(s_1; c_1); (s_2; c_3); (s_3; c_2); (s_4; c_2)\}$. This matching satis es individual rationality.

In some school districts, each student gets a priority at her neighborhood school, as in this example. In the absence of other types of priorities, neighborhood priority guarantees that SPDA satis es individual rationality.

3.2. Balanced Exchange. For an interdistrict school choice program, maintaining a balance of students incoming from and outgoing to other districts is important. To formalize this idea, we say that a mechanism satis es the *balanced-exchange* policy if the number of students that a district gets from the other districts and the number of students that the district sends to the others are the same for every district and for every pro le of student preferences. Since district choice rules are acceptant, every student is matched with a school under SPDA. Therefore, for SPDA, this policy is equivalent to the requirement that the number of students assigned to a district must be equal to the number of students from that district.

The balanced-exchange policy is important because the funding that a district gets depends on the number of students it serves. Therefore, an interdistrict school choice program may not be sustainable if SPDA does not satisfy the balanced-exchange policy. For achieving this policy goal, the following condition on admissions rules proves important.

De nition 2. A matching X is *rationed* if, for every district d, it does not assign strictly more students to the district than the number of students whose home district is d. A district admissions rule is *rationed* if it chooses a rationed matching from any fromhan rationed. Conversely, when there exists one district with an admissions rule that fails to be rationed, then we can construct student preferences such that this district is matched with strictly more students than the number of students from the district in SPDA, which means that the outcome does not satisfy the balanced-exchange policy.

Now we illustrate SPDA when district admissions rules are rationed.

Example 3. Consider the problem in Example 1. Recall that in this problem, the SPDA outcome is $\{(s_1,c_2),(s_2,c_3),(s_3,c_1),(s_4,c_2)\}$. Since there are three students matched with district d_1 and there are only two students from that district, SPDA does not satisfy the balanced-exchange policy. This is consistent with Theorem 2 because the admissions rule of district d_1 is not rationed. In particular, $Ch_{d_1}(\{(s_1,c_2),(s_3,c_1),(s_4,c_2)\}) = \{(s_1,c_2),(s_3,c_1),(s_4,c_2)\}$, so district d_1 accepts more students than the number of students from there given a matching that is feasible for students.

Suppose that we modify the admissions rule of district d_1 as follows. If the district chooses a contract associated with school c_1, then at most one contract associated with school c_2 is chosen. Therefore, the district never chooses more than two contracts, which is the number of students from there. Therefore, the updated admissions rule is rationed.[18] With this change, it is easy to check that the SPDA outcome is $\{(s_1,c_2),(s_2,c_3),(s_3,c_1),(s_4,c_3)\}$, which satisfies the balanced-exchange policy.

An implication of Theorems 1 and 2 is that SPDA is guaranteed to satisfy individual rationality and the balanced-exchange policy if, and only if, each district's admissions rule respects the initial matching and is rationed.

3.3. Diversity. The third policy goal we consider is that of diversity. More specifically, we are interested in how to ensure that there is enough diversity across districts so that the student composition in terms of demographics does not vary too much from district to district.

We are mainly motivated by a program that is used in the state of Minnesota. State law in Minnesota identifies racially isolated (relative to one of their neighbors) school districts and requires them to be in the Achievement and Integration (AI) Program. The goal is to increase the racial parity between neighboring school districts. We first introduce a diversity policy in the spirit of this program: Given a constant $\alpha \in [0,1]$, we say that a mechanism satisfies the α-*diversity policy* if for all preferences, districts d and d', and type t, the difference between the ratios of type-t students in districts d and d' is not more than α. We interpret α to be the maximum ratio difference tolerated under the diversity policy; for instance, $\alpha = 0.2$ for Minnesota.

[18] In Appendix B.3, we construct a class of rationed district admissions rules that includes this admissions rule as a special case. These admissions rules are feasible and acceptant, and they have completions that satisfy substitutability and LAD.

We study admissions rules such that SPDA satisfies the α-diversity policy when there is interdistrict school choice. Since this policy restricts the number of students across districts, a natural starting point is to have type-specific ceilings at the district level. However, it turns out that type-specific ceilings at the district level may result in district admissions rules resulting in no stable matchings (see Theorem 9 in Appendix A.2).

Since there is an incompatibility between district-level type-specific ceilings and the existence of a stable matching, we impose type-specific ceilings at the school level as follows.

Definition 3. A district admissions rule Ch_d has a *school-level type-specific ceiling* of q_c^t at school c for type-t students if the number of type-t students admitted cannot exceed this ceiling. More formally, for any matching X that is feasible for students,

$$|\{x \in Ch_d(X) \mid \tau(s(x)) = t, c(x) = c\}| \leq q_c^t.$$

Note that district admissions rules typically violate acceptance once school-level type-specific ceilings are imposed. This is because a student can be rejected from a set that is feasible for students even when the number of applicants to each school is weakly smaller than its capacity and the number of applicants to the district is weakly smaller than the number of students from that district. Given this, we define a weaker version of the acceptance assumption as follows.

Definition 4. A district admissions rule Ch_d that has school-level type-specific ceilings is *weakly acceptant* if, for any contract x associated with a type-t student and district d and matching X that is feasible for students, if x is rejected from X, then at $Ch_d(X)$,

- the number of students assigned to school $c(x)$ is equal to $q_{c(x)}$, or
- the number of students assigned to district d is at least k_d, or
- the number of type-t students assigned to school $c(x)$ is at least $q_{c(x)}^t$.

In other words, a student can be rejected from a set that is feasible for students only when one of these three conditions is satisfied.

In SPDA, a student may be left unassigned due to school-level type-specific ceilings even when district admissions rules are weakly acceptant. To make sure that every student is matched, we make the following assumption.

Definition 5. District admissions rules $(Ch_d)_{d \in D}$ *accommodate unmatched students* if for any student o1y4 students because an unmatched student's application to her initial school is always accepted. Lemma 2 in Appendix D shows that when district admissions rules accommodate unmatched students, every student is matched to a school in SPDA.

In general, accommodation of unmatched students may be in con ict with type-speci c ceilings because there may not be enough space for a student type when ceilings are small for this type. To avoid this, we assume that type-speci c ceilings are high enough so that $(Ch_d)_{d2D}$ accommodate unmatched students.[19]

Both of these optimization problems belong to a special class of linear-programming problems called a minimum-cost flow problem, and many computationally efficient algorithms to solve it are known in the literature.[20] A straightforward but important observation is that $\hat{p}^t_d$ (resp. $\hat{q}^t_d$) is exactly the lowest (resp. highest) number of type-t students who can be matched to district d in a legitimate matching (Lemma 3 in Appendix D). Given this observation, we call $\hat{p}^t_d$ the *implied floor* and $\hat{q}^t_d$ the *implied ceiling*.

Now we are ready to state the main result of this section.

Theorem 3. Suppose that each district admissions rule has school-level type-specific ceilings and is rationed and weakly acceptant. Moreover, suppose that the district admissions rules accommodate unmatched students. SPDA satisfies the diversity policy if, and only if, $\hat{q}^t_d/k_d \geq \hat{p}^t_{d'}/k_{d'}$ for every type t and districts d, d' such that $d \neq d'$.

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Consider the case where the mechanism produces matching X at the above student preference pro le. Suppose student s_3

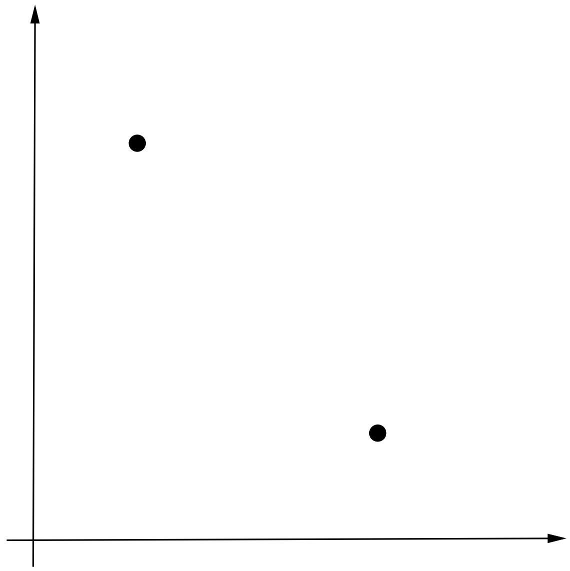


Figure 2. Illustration of M-convexity

Example 5. Consider the problem and the set of distributions Ξ defined in Example 4. We show that Ξ is not M-convex. Recall matchings X and X' in that example. By construction, both X and X' satisfy the policy goal Ξ. Furthermore, $\xi^{t_1}_{c_3}(X) = 1 > 0 = \xi^{t_1}_{c_3}(X')$ because (i) school c_3 is matched with student s_4 at X, whose type is t_1, while (ii) school c_3 is matched with student s_5 at X', whose type is $t_2 \neq t_1$. If the set of distributions Ξ is M-convex, there exist a school c and a type t such that $\xi^t_c(X) < \xi^t_c(X')$ and $\xi(X) - \chi_{c_3,t_1} + \chi_{c,t}$ is in Ξ. Because each school's capacity is one, and at matching X all schools have filled their capacities, this means that the only candidate for (c, t) satisfying the above condition is such that $c = c_3$. But the only nonzero $\xi^t_{c_3}(X')$ is for $t = t_2$ (corresponding to s_5 matched with c_3 at X'), and $\xi(X) - \chi_{c_3,t_1} + \chi_{c_3,t_2}$ does not satisfy the policy goal because district d_1's ceiling for type t_2 is violated (note $\xi^{t_2}_{c_2}(X) = 1$ because student s_2 is matched with c_2 at X.)

The above argument implies that $\Xi \cap \Xi^0$ is not M-convex either. To see this, note that both $\xi(X)$ and $\xi(X')$ are in $\Xi \cap \Xi^0$ because all students are matched. Because we have shown that no distribution of the form $\xi(X) - \chi_{c_3,t_1} + \chi_{c,t}$ is in Ξ, by set inclusion relation $\Xi \cap \Xi^0 \subseteq \Xi$, there is no distribution of the form $\xi(X) - \chi_{c_3,t_1} + \chi_{c,t}$ in $\Xi \cap \Xi^0$ either.

$c, c' \in \mathcal{C}$, and $(c_0, t)\ \tilde{P}_s\ (c, t')$ for any $c \in \mathcal{C}$ and $t' \in \mathcal{T}$ such that $t' \neq t$. That is, $\tilde{P}_s$ is a preference order over school-type pairs that ranks the school-type pairs in which the type is t in the same order as in P_s, while finding all school-type pairs specifying a different

The main result of this section is as follows.

Theorem 5. Suppose that the initial matching satisfies the policy goal. If $\cap\ {}^0$ is M-convex, then TTC satisfies the policy goal, constrained efficiency, individual rationality, and strategy-proofness.

The assumption that the initial matching satisfies the policy goal is necessary for the result: Consider student preferences such that each student's highest-ranked school is her initial school. Then the initial matching is the unique individually rational matching. Therefore, if there exists a mechanism with the desired properties, then the outcome at this preference profile has to be the initial matching. Hence, we need the assumption that the initial matching satisfies the policy goal to have such a mechanism.

To see one of the implications of this theorem, suppose that the policy goal is such that no school is matched with more students than its capacity. In that case, if is M-convex, then TTC satisfies the desirable properties.

Corollary 1. Suppose that the policy goal is such that for every $\in$ and $c \in C$, $\sum_t {}^t_c\ q_c$. Furthermore, suppose that the initial matching satisfies. If is M-convex, then TTC satisfies the policy goal , constrained efficiency, individual rationality, and strategy-proofness.

In the proof of this corollary, we show that when is M-convex and no distribution in assigns more students to a school than its capacity, then $\cap\ {}^0$ is also M-convex. Therefore, the corollary follows directly from Theorem 5.

Next we illustrate TTC with an example.

Example 6. Consider a problem with two school districts, d_1 and d_2. District d_1 has school c_1 with capacity three and school c_2 with capacity two. District d_2 has school c_3 with capacity two and school c_4 with capacity one. There are seven students: students s_1, s_2, s_3, and s_4 are from district d_1 and have type t_1, whereas students s_5, s_6, and s_7 are from district d_2 and have type t_2. The initial matching is $\{(s_1; c_1); (s_2; c_1); (s_3; c_2); (s_4; c_2); (s_5; c_3); (s_6; c_3); (s_7; c_4)\}$. Student preferences are as follows.

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	P_{s_5}	P_{s_6}	P_{s_7}
c_2	c_3	c_4	c_2	c_1	c_4	c_2
c_3	c_1	c_2	c_3	c_2	c_1	c_3
$\vdots$	$\vdots$	$\vdots$	c_1	c_3	c_3	c_1
			c_4	c_4	c_2	c_4

In addition to the school capacities, there is only one additional constraint that school c_1 cannot have more than one type-t_2 student. As we show in the proof of Corollary 2, the set of distributions that satisfy this policy goal and the requirement that every student is matched is an M-convex set. Therefore, by Theorem 5, TTC satisfies constrained efficiency, individual rationality, strategy-proofness, and the policy goal.

To run TTC, we use a master priority list. Suppose that the master priority list ranks students as follows: $s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_5 \succ s_6 \succ s_7$.

At Step 1 of TTC, there are eight school-type pairs. Consider $(c_1; t_1)$. Initially, students s_1 and s_2 are matched with it, so they are both permissible to this pair. We use the master priority list to rank them, so s_1 gets the highest priority at $(c_1; t_1)$. Therefore, $(c_1; t_1)$ points to s_1. Now consider $(c_1; t_2)$. Initially, it does not have any students because there is no type-t_2 student assigned to c_1 in the original problem. Furthermore, s_1 is permissible to $(c_1; t_2)$ because she can be removed from $(c_1; t_1)$ and a type-t_2 student can be assigned to $(c_1; t_2)$ without violating the school capacities or the policy goal. Therefore, $(c_1; t_2)$ points to s_1 as well, who gets a higher priority than the other permissible students because of the master priority list. The rest of the pairs also point to the highest-priority permissible students. Each student points to the highest ranked school-type pair of the same type as shown in Figure 3A. There is only one cycle: $s_7 \to (c_2; t_2) \to s_3 \to (c_4; t_1) \to s_7$. Therefore, s_7 is matched with $(c_2; t_2)$ and s_3 is matched with $(c_4; t_1)$.

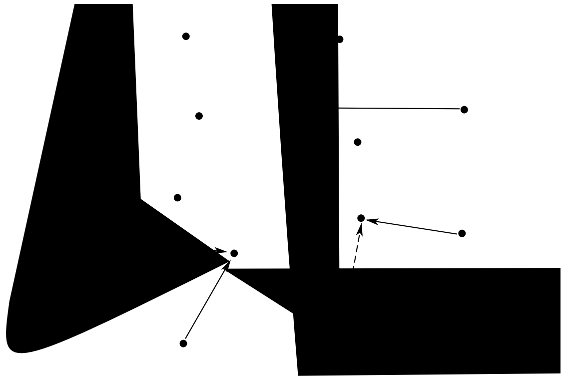
At Step 2, there are six remaining school-type pairs: There are no permissible students for $(c_4; t_1)$ and $(c_4; t_2)$ because c_4 has a capacity of one and it is already assigned to s_3. Each remaining school-type pair points to the highest-ranked remaining permissible student. Each student points to the highest-ranked remaining school-type pair (see Figure 3B). There is only one cycle: $s_4 \to (c_2; t_1) \to s_4$. Hence, s_4 is assigned to $(c_2; t_1)$.

The algorithm ends in five steps. Steps 3 and 4 are also shown in Figure 3. In Step 5, s_2 points to $(c_1; t_1)$, which points back to the student. The outcome of the algorithm is

$$\{(s_1; c_3); (s_2; c_1); (s_3; c_4); (s_4; c_2); (s_5; c_1); (s_6; c_3); (s_7; c_2)\}:$$

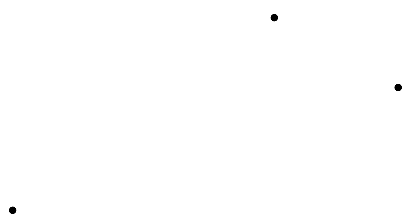
It can be easily seen that the distribution associated with this matching satisfies the policy goal because no school has more students than its capacity and c_1 has only one type-t_2 student.

Sometimes it may be more convenient to describe a policy goal using a real-valued function rather than a set of distributions. The interpretation is that the policy function measures how satisfactory the distribution is in terms of the policy goal. To formalize this alternative approach let $f : \mathbb{Z}_+^{|C||T|} \to \mathbb{R}$ be a function on distributions such that $f(\xi) \geq f(\xi')$ means that distribution ξ satisfies the policy at least as well as distribution ξ'. Let $\lambda \in \mathbb{R}$ be a constant. Consider the following $(f; \lambda)$ policy: $\Xi(f; \lambda) \equiv \{\xi \in \mathbb{Z}_+^{|C||T|} | f(\xi) \geq \lambda\}$. Note that the initial matching $\tilde{X}$ satisfies the $(f; \lambda)$-policy if, and only if, $f(\xi(\tilde{X})) \geq \lambda$.

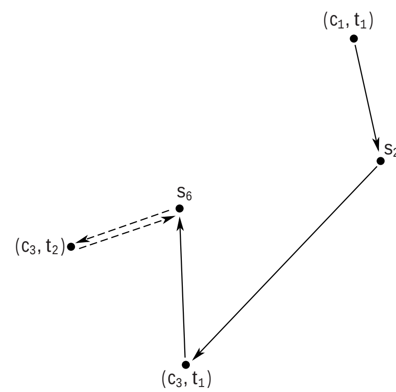

[redacted]

(a) Step 1 of TTC

(b) Step 2 of TTC



(c) Step 3 of TTC



(d) Step 4 of TTC

Figure 3. The first four steps of TTC. In each step, there is only one cycle, which is represented by the dashed lines.

To see why this theorem holds, recall that by Lemma 1, (f;) \ ^{0} is M-convex. Furschool level. Taken together, these results inform policy makers about what kinds of diversity policies are compatible with the other desiderata.

One possible shortcoming of Corollary 2 is that the result holds under the assumption that the initial matching satisfies the school-level diversity policy. This may be undesirable given that often diversity policies are implemented because schools or districts are regarded as insufficiently diverse, as in the case of the diversity law in Minnesota. In such a setting, a potential diversity requirement can be that the diversity should not decrease as a result of interdistrict school choice according to a diversity measure f. Such a consideration can be formally described as the $(f, \xi(\bar{X}))$-policy, $\Xi(f, \xi(\bar{X}))$. The next corollary establishes a positive result for a $\Xi(f, \xi(\bar{X}))$-policy where the diversity is measured via the "Manhattan distance" to an ideal point.

Corollary 3. Let $\hat{\xi} \in \Xi_0$ be an ideal distribution and $f(\xi) \equiv -\sum_{c,t} |\xi_c^t - \hat{\xi}_c^t|$ be the policy function. Then TTC satisfies $(f, \xi(\bar{X}))$-policy, constrained efficiency, individual rationality, and strategy-proofness.

Note that the initial matching $\bar{X}$ always satisfies $(f, \xi(\bar{X}))$-policy. Furthermore, we show that the policy function f is pseudo M-concave. Therefore, this corollary follows from Theorem 6. More generally, when the diversity is measured by a pseudo M-concave function, then the TTC outcome is as diverse as the initial matching. Furthermore, TTC also satisfies the other desirable properties.

Next, we study the balanced-exchange policy introduced in Section 3.2. We establish that the balanced-exchange policy is represented by a distribution that satisfies M-convexity. This implies the following result.

Corollary 4. TTC satisfies the balanced-exchange policy, constrained efficiency, individual rationality, and strategy-proofness.

One of the advantages of our approach is that M-convexity of a set and pseudo M-concavity of a function are so general that a wide variety of policy goals satisfy them, and that it is likely to be applicable for policy goals that one may encounter in the future. To highlight this point, we consider imposing the diversity and balanced-exchange policies at the same time. More specifically, define a set of distributions $\{\xi \mid q_c^t \le \xi_c^t \le p_c^t$ for all c and t; $q_c \le \sum_t \xi_c^t$ for all c; and $\sum_t \sum_{c:d(c)=d} \xi_c^t = k_d$ for all $d\}$ and call it the *combination of balanced-exchange and school-level diversity policies*. This is the set of distributions that satisfy both the school-level floors and ceilings and the balanced-exchange requirement. We can establish this set is M-convex, implying the following result.

Corollary 5. Suppose that the initial matching satisfies the combination of balanced exchange and school-level diversity policies. Then TTC satisfies the combination of balanced exchange and school-level diversity policies, constrained efficiency, individual rationality, and strategy-proofness.

In the proof, we show that the combination of balanced exchange and school-level diversity policies satis es M-convexity. In general, the intersection of two M-convex sets need not be M-convex.[22] Therefore, the M-convexity of the combination of balanced exchange and school-level diversity policies does not ore,hevexity of the oalancednxchange policiy-125(pnd)-285) of schools). Given that the existing literature has not studied interdistrict school choice, we envision that many policy goals await study within our framework.

While our paper is primarily theoretical and aimed at proposing a general framework to study interdistrict school choice, the main motivation comes from applications to actual programs such as Minnesota's AI program. Given this motivation, it would be interesting to study interdistrict school choice empirically. For instance, evaluating how well the existing programs are doing in terms of balanced exchange, student welfare, and diversity, and how much improvement could be made by a conscious design based on theories such as the ones suggested in the present paper, are important questions left for future work. In addition, implementation of our designs in practice would be interesting. Doing so may, for instance, shed new light on the tradeo between SPDA and TTC, which has been stud-

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with the home district is the same as the relative ranking in the original preferences. Importantly, in this setting, the initial matching is not fixed but is determined by student preferences and district admissions rules. In such a setting, we characterize district admissions rules which guarantee that no student is hurt from interdistrict school choice.

The next property of district admissions rules proves to play a crucial role to achieve this policy.

Definition 8. A district admissions rule Ch_d *favors own students* if for any matching X that is feasible for students,

$$Ch_d(X) \supseteq Ch_d(\{x \in X | d(s(x)) = d\}).$$

When a district admissions rule favors own students, any contract that is chosen from a set of contracts associated with students from a district is also chosen from a superset that includes additional contracts associated with students from the other districts. Roughly, this condition requires that, under interdistrict school choice, a district prioritizes its own students that it used to admit over students from the other districts (even though an out-of-district student can still be admitted when a student from the district is rejected).

The following result shows that this is exactly the condition which guarantees that interdistrict school choice weakly improves the outcome for every student.

Theorem 8. Every student weakly prefers the DA outcome under interdistrict school choice to the DA outcome under intradistrict school choice for all student preferences if, and only if, each district's admissions rule favors own students.

In the proof, we show that in the intradistrict school choice the SPDA outcome can alternatively be produced by an interdistrict school choice model where students rank contracts with all districts and districts have modified admissions rules: For any set of contracts X, each district d chooses the following contracts: $Ch_d(\{x \in X | d(s(x)) = d\})$. Since the original district admissions rules favor own students, the chosen set under the modified admissions rule is a subset of $Ch_d(X)$ when X is feasible for students. Then the conclusion that students receive weakly more preferred outcomes in interdistrict school choice than in intradistrict school choice follows from a comparative statics property of SPDA that we show (Lemma 5).[24] To show the "only if" part, when there exists a district admissions rule that fails to favor own students, we construct preferences of students such that interdistrict school choice makes at least one student strictly worse off than intradistrict school choice.

A.2. District-level Type-specific Ceilings. In this section, we show the incompatibility of type-specific ceilings at the district level with the existence of a stable matching.

[24] We cannot use the comparative statics result of Yenmez (2018) because in our setting $Ch_d(X)$

Definition 9. A district admissions rule Ch_d has a *district-level type-specific ceiling* of q_d^t for type-t students if the number of type-t students admitted cannot exceed this ceiling. More formally, for any matching X that is feasible for students,

$$|\{x \in Ch_d(X) \mid \tau(s(x)) = t\}| \le q_d^t.$$

Note that, as in the case of school-level type-specific ceilings, district admissions rules do not necessarily satisfy acceptance once district-level type-specific ceilings are imposed. We define a weaker version of the acceptance assumption as follows.

Definition 10. A district admissions rule Ch_d that has district-level type-specific ceilings is *weakly acceptant* if, for any contract x associated with a type-t student and district d and matching X that is feasible for students, if x is rejected from X, then at $Ch_d(X)$,

- the number of students assigned to school $c(x)$ is equal to $q_{c(x)}$, or
- the number of students assigned to district d is at least

Since all school admissions rules satisfy substitutability and LAD, so does Ch_d^0.

All of the assumptions on school admissions rules stated in Claims 1, 2, and 3 are satis- ed when school admissions rules are *responsive*: each school has a ranking of contracts associated with itself and, from any given set of contracts, each school chooses contracts with the highest rank until the capacity of the school is full or there are no more contracts left. Responsive admissions rules satisfy substitutability and LAD. Furthermore, for every school c_i, $|Ch_{c_i}(X)| = \min\{q_{c_i}; |X_{c_i}|\}$.[25] By the claims stated above, when school admissions rules are responsive, district admissions rule Ch_d is feasible and acceptant, and it has a completion that satis es substitutability and LAD.

Based on these results, we provide examples of district admissions rules that further satisfy the additional assumptions considered in di erent parts of our paper.

B.2. District Admissions Rules Satisfying the Assumptions in Theorem 1. We use the district admissions rule construction above and we further specify each school's admissions rule. Each school has a responsive admissions rule. If a student is initially matched with a school, then her contract with this school is ranked higher than contracts of students who are F26 4P4831 Td [(25)]TJ/F8 -17.309 Td [(2.989 4]TJ -11.955 -17.(2.2o03-mF26 411.6

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Proof. To show acceptance, suppose that matching X is feasible for students and x 2 X_d n $Ch_d(X)$. There exists i n such that $c_i = c(x)$. Since X is feasible for students, x 2 X n $Y_{i\ 1}$ where $Y_{i\ 1}$ is the set of all contracts in X associated with students who are chosen by schools $c_1; : : : ; c_{i\ 1}$. Because x 2 X_d n $Ch_d(X)$, x is not chosen by c_i. Then, by construction, either c_i lls its capacity or the district admits k_d students, which implies that Ch_d is acceptant.

Claim 6. District admissions rule Ch_d has a completion that satis es substitutability and LAD.

Proof. First, we construct a completion of Ch_d. De ne the following district admissions rule: given a set of contracts X, when it is the turn of a school, it chooses from all the contracts in X

A reserve for a student type at a school c guarantees space for this type at school c. Therefore, when a student is unmatched at a feasible matching and the reserve for her type is not yet filled at a school, the district will accept this student at that school if she applies to it.

Claim 7. Suppose that districts have admissions rules with reserves such that $\sum_c r_c^t = k^t$ for every type t. Then district admissions rules accommodate unmatched students.

Proof. Suppose that student s is unmatched at a feasible matching X. Let t be the type of student s. Then there exists a school c such that the number of type-t students in c at X

Proof. For any set of contracts X, school c, and type t, let X_c^t denote the set of all contracts in X that are associated with school c and type-t students.

Consider the construction of Ch_d above, but modify it by not removing contracts of students who are chosen previously. Denote this district admissions rule by Ch_d^0. To show that Ch_d^0 is a completion of Ch_d, consider a set of contracts X and suppose f contracts has to be the case thatx

B.5. District Admissions Rules Satisfying the Assumptions in Theorem 8. Consider the district admissions rule construction in Appendix B.1. In this example, let each school use a priority ranking in such a way that all contracts of students from district d are ranked higher than the other contracts.

Claim 9. District admissions rule Ch_d favors own students.

Proof. Suppose that X is feasible for students. When it is the turn of school c_i, it considers X_{c_i}. Therefore, $Ch_d(X) = Ch_{c_1}(X_{c_1}) \cup \cdots \cup Ch_{c_k}(X_{c_k})$. Furthermore, $Ch_{c_i}(X_{c_i}) \subseteq Ch_{c_i}(\{x \in X_{c_i} | d(s(x)) = d\})$ by construction. Taking the union of all subset inclusions yields $Ch_d(X) \subseteq Ch_d(\{x \in X_d | d(s(x)) = d\})$. Therefore, Ch_d favors own students.

Appendix C. An Example for Diversity

In this section, we provide an example in which the conditions on the admissions rules stated in Theorem 3 are satisfied and, therefore, SPDA satisfies the diversity policy.

Consider a problem with two school districts, d_1 and d_2. District d_1 has school c_1 with capacity three and school c_2 with capacity two. District d_2 has school c_3 with capacity two and school c_4 with capacity one. There are seven students: students s_1, s_2, s_3, and s_4 are from district d_1, whereas students s_5, s_6, and s_7 are from district d_2. Students s_1, s_5, s_6, and s_7 have type t_1 and s_2, s_3, and s_4 have type t_2. To construct district admissions rules that satisfy the properties stated in Theorem 3, let us first specify type-specific ceilings and calculate implied floors and implied ceilings. Suppose that

$$q_{c_1}^{t_1} = 1;\ q_{c_1}^{t_2} = 2;\ q_{c_2}^{t_1} = 1;\ q_{c_2}^{t_2} = 1;$$
$$q_{c_3}^{t_1} = 2;\ q_{c_3}^{t_2} = 1;\ q_{c_4}^{t_1} = 1;\ q_{c_4}^{t_2} = 1.$$

These yield the following implied floors,[26]

$$\hat{p}_{d_1}^{t_1} = 1;\ \hat{p}_{d_1}^{t_2} = 2;\ \hat{p}_{d_2}^{t_1} = 2;\ \hat{p}_{d_2}^{t_2} = 0;$$

and implied ceilings

$$\hat{q}_{d_1}^{t_1} = 2;\ \hat{q}_{d_1}^{t_2} = 3;\ \hat{q}_{d_2}^{t_1} = 3;\ \hat{q}_{d_2}^{t_2} = 1.$$

[26] To see this, note that there cannot be zero type-t_1 students in d_1 (otherwise not all type-t_1 students can be matched since there are only three spaces available for type-t_1 students in d_2). If there is one type-t_1 student in d_1, there has to be three type-t_1 students in d_2, which implies there cannot be any type-t_2 students in d_2, and this implies there will be three type-t_2 students in d_1. If there are two type-t_1 students in d_1, there have to be two type-t_2 students in d_2, which implies there is one type-t_2 student in d_2, and this implies there will be two type-t_2 students in d_1. By noting these minimum and maximum numbers, we obtain the implied reserves and implied ceilings accordingly. These bounds are achievable because it is feasible to have (i) one type-t_1 student in d_1, three type-t_1 students in d_2, zero type-t_2 students in d_2, and three type-t_2 students in d_1, and (ii) two type-t_1 students in d_1, two type-t_2 students in d_2, one type-t_2 student in d_2, and two type-t_2 students in d_1.

For any type t and two districts d and d', denote $\hat{q}^t_d/k_d \quad \hat{p}^t_{d'}/k_{d'}$ by $^t_{d;d'}$. Using the implied oors and ceilings above, we get:

$$
\begin{aligned}
{}^{t_1}_{d_1;d_2} &= 2/4 \quad 2/3 = \quad 1/6;\\
{}^{t_1}_{d_2;d_1} &= 3/3 \quad 1/4 = 3/4;\\
{}^{t_2}_{d_1;d_2} &= 3/4 \quad 0/3 = 3/4; \text{ and}\\
{}^{t_2}_{d_2;d_1} &= 1/3 \quad 2/4 = \quad 1/6.
\end{aligned}
$$

Hence, these type-speci c ceilings satisfy the condition stated in Theorem 3 that $^t_{d;d'}$ for $= 0.75$.

We construct district admissions rules that have type-speci c ceilings, accommodate

the first step. Therefore, student s is matched with a strictly less preferred school than her initial school, which implies that SPDA does not satisfy individual rationality.

Proof of Theorem 2. We first prove that if each district admissions rule is rationed, then SPDA satisfies the balanced-exchange policy. Let X be the matching produced by SPDA for a given preference profile.

We begin by showing that each student must be matched with a school in X. Suppose, for contradiction, that student s is unmatched. Since X is a stable matching, every contract $x = (s; d; c)$ associated with the student is rejected by the corresponding district, i.e., $x \notin Ch_d(X \cup \{x\})$. Otherwise, student s and district d would like to match with each other using contract x, contradicting the stability of matching X. Since $X \cup \{x\}$ is feasible for students, acceptance implies that, for each district d, either every school in the district is full or that the district has at least k_d students at matching X. Both of them imply that the district has at least k_d students in matching X since the sum of the school capacities in district d is at least k_d. But this is a contradiction to the assumption that student s is unmatched since the existence of an unmatched student implies that there is at least one district d such that the number of students in X_d is less than k_d. Therefore, all students are matched in X.

Because X is the outcome of SPDA, it is feasible for students. Therefore, because district admissions rules are rationed, the number of students in district d cannot be strictly more than k_d for any district d. Furthermore, since every student is matched, the number of students in district d must be exactly k_d (because, otherwise, at least one student would have been unmatched.) As a result, SPDA satisfies the balanced-exchange policy.

Next, we prove that if at least one district's admissions rule fails to be rationed, then there exists a student preference profile under which SPDA does not satisfy the balanced-exchange policy. Suppose that there exist a district d and a matching X, which is feasible for students, such that $|Ch_d(X)| > k_d$. Consider a feasible matching X' such that (i) all students are matched, (ii) $X'_d = Ch_d(X)$, and (iii) for every district $d' \neq d$, $|X'_{d'}| \leq k_{d'}$. The existence of such X' is guaranteed since every district has enough capacity to serve its students (i.e., for every district d', $\sum_{c:d(c)=d'} q_c \geq k_{d'}$), and $|Ch_d(X)| > k_d$. Now, consider any student preferences, where every student likes her contract in X' the most.

We show that SPDA stops in the first step. For district $d' \neq d$, $X'_{d'}$ is feasible and the number of students matched to which implies $Ch_d(Ch_d(X)) = Ch_d(X)$. As a result, $Ch_d(X'_d) = X'_d$. Therefore, SPDA stops at the first step since no contract is rejected.

Since SPDA stops at the first step, the outcome is matching X'. But X' fails the balanced-exchange policy because $|X'_d| = |Ch_d(X)| > k_d$.

Proof of Theorem 3. To prove this result, we provide the following lemmas.

Lemma 2.

To prove the above claim, assume for contradiction that there exists $X \in M_2$ such that $\xi^t_d(X) \neq \xi^t_d(\hat{X})$. By Lemma 3, $\xi^t_d(X) \neq \xi^t_d(\hat{X})$ implies that $\xi^t_d(X) < \xi^t_d(\hat{X})$. Then there exists c with $d(c) = d$ such that $\xi^t_c(X) < \xi^t_c(\hat{X})$. Consider the following procedure.

Step 0: Initialize by setting $(t_1; c_1) := (t; c)$. Note that $\xi^{t_1}_{c_1}(X) < \xi^{t_1}_{c_1}(\hat{X})$ by definition of c.

Step $i \ge 1$: We begin with $(t_i; c_i)$. Note that (by assumption for $i = 1$, and as shown later for $i \ge 2$), $\xi^{t_i}_{c_i}(X) < \xi^{t_i}_{c_i}(\hat{X})$. Denote $d_i = d(c_i)$. Now,

(1) Suppose that there exists $i' < i$ such that either (i) $c_{i'} = c_i$ or (ii) $\xi^{t_i}_{c_i}(X) < q_{c_i}$ and $d(c_{i'}) = d(c_i)$. If such an index i' exists, then set $(t_{i+1}; c_i) := (t_{i'+1}; c_{i'})$.

(2) Suppose not. Then, if there exists $t' \in T$ such that $\xi^{t'}_{c_i}(X) > \xi^{t'}_{c_i}(\hat{X})$, then set $t_{i+1} := t'$ and $c_i := c_i$.

(3) If not, then note that $\sum_{\tilde{t} \in T} \xi^{\tilde{t}}_{c_i}(X) < q_{c_i}$.[28] Also note that there exists a type-school pair (t', c') with $c' \neq c_i$ such that $\xi^{t'}_{c'}(X) > \xi^{t'}_{c'}(\hat{X})$ and $d(c') = d_i$ because $\sum_{\tilde{c}: d(\tilde{c}) = d_i; \tilde{t} \in T} \xi^{\tilde{t}}_{\tilde{c}}(X) = \sum_{\tilde{c}: d(\tilde{c}) = d_i; \tilde{t} \in T} \xi^{\tilde{t}}_{\tilde{c}}(\hat{X}) = k_{d_i}$.

(a) If $t' = t_i$, then let $\tilde{X}$ be a matching such that

$$\xi^{\tilde{t}}_{\tilde{c}}(\tilde{X}) = \begin{cases} \xi^{t_i}_{c_i}(X) + 1 & \text{for } (\tilde{t}, \tilde{c}) = (t_i; c_i); \\ \xi^{t'}_{c'}(X) - 1 & \text{for } (\tilde{t}, \tilde{c}) = (t_i; c'); \\ \xi^{\tilde{t}}_{\tilde{c}}(X) & \text{otherwise.} \end{cases}$$

Note that $\tilde{X} \in M_1$.[29] Also, by construction, $\sum_{\tilde{t}, \tilde{c}} |\xi^{\tilde{t}}_{\tilde{c}}(\tilde{X}) - \xi^{\tilde{t}}_{\tilde{c}}(\hat{X})| = \sum_{\tilde{t}, \tilde{c}} |\xi^{\tilde{t}}_{\tilde{c}}(X) - \xi^{\tilde{t}}_{\tilde{c}}(\hat{X})| - 2 < \sum_{\tilde{t}, \tilde{c}} |\xi^{\tilde{t}}_{\tilde{c}}(X) - \xi^{\tilde{t}}_{\tilde{c}}(\hat{X})|$, which contradicts the assumption that $X \in M_2$.

(b) Therefore, suppose that $t' \neq t_i$ and let $t_{i+1} := t'$ and $c_i := c'$.

(4) The pair $(t_{i+1}; c_i)$ created above satisfies $\xi^{t_{i+1}}_{c_i}(X) > \xi^{t_{i+1}}_{c_i}(\hat{X})$, so there exists $c' \in C$ such that $\xi^{t_{i+1}}_{c'}(X) < \xi^{t_{i+1}}_{c'}(\hat{X})$. Set $c_{i+1} = c'$. Note that $\xi^{t_{i+1}}_{c_{i+1}}(X) < \xi^{t_{i+1}}_{c_{i+1}}(\hat{X})$.

We follow the procedure above to define $(t_1; c_1); (t_2; c_1); (t_2; c_2); (t_3; c_2); (t_3; c_3)$, and so forth. Because the set T is finite, we have i and $j > i$ with $t_i = t_j$. Consider the smallest j with this property (note that given such j, i is uniquely identified). Now, let $\tilde{X}$ be a

[28] A proof of this fact is as follows. By an earlier argument, $\xi^{t_i}_{c_i}(X) < \xi^{t_i}_{c_i}(\hat{X})$. Moreover, by assumption $\xi^{\tilde{t}}_{c_i}(X) \le \xi^{\tilde{t}}_{c_i}(\hat{X})$ for every $\tilde{t} \in T$. Therefore, $\sum_{\tilde{t} \in T} \xi^{\tilde{t}}_{c_i}(X) < \sum_{\tilde{t} \in T} \xi^{\tilde{t}}_{c_i}(\hat{X}) \le q_{c_i}$.

[29] A proof of this fact is as follows. Because $\sum_{\tilde{t} \in T} \xi^{\tilde{t}}_{c_i}(X) < q_{c_i}$, $\sum_{\tilde{t} \in T} \xi^{\tilde{t}}_{c_i}(\tilde{X}) = \sum_{\tilde{t} \in T} \xi^{\tilde{t}}_{c_i}(X) + 1 \le q_{c_i}$. For every $\tilde{c} \neq c_i$, $\sum_{\tilde{t} \in T} \xi^{\tilde{t}}_{\tilde{c}}(\tilde{X}) \le \sum_{\tilde{t} \in T} \xi^{\tilde{t}}_{\tilde{c}}(X) \le q_{\tilde{c}}$. Thus, all school capacities are satisfied. For all $\tilde{c}, \tilde{t}$, $\xi^{\tilde{t}}_{\tilde{c}}(\tilde{X}) \le \max\{\xi^{\tilde{t}}_{\tilde{c}}(X); \xi^{\tilde{t}}_{\tilde{c}}(\hat{X})\} \le q^{\tilde{t}}_{\tilde{c}}$ by construction, so all type-specific ceilings are satisfied. And $\sum_{\tilde{t} \in T; \tilde{c} \in C} \xi^{\tilde{t}}_{\tilde{c}}(\tilde{X}) = \sum_{\tilde{t} \in T} \xi^{\tilde{t}}_{\tilde{c}}(X)$ by definition of $\tilde{X}$, so $\tilde{X}$ is a legitimate matching. Finally, $\xi^{\tilde{t}}_{\tilde{d}}(\tilde{X}) = \xi^{\tilde{t}}_{\tilde{d}}(X)$ for every $\tilde{t}$ and $\tilde{d}$, so $\tilde{X} \in M_1$.

matching such that

$$\xi^{t}_{c}(\bar X) = \begin{cases} \xi^{t_k}_{c_k}(X)+1 & \text{for } (\bar t,\bar c)=(t_k,c_k) \text{ for any } k\in\{i,i+1,\dots,j-1\};\\ \xi^{t_{k+1}}_{c_k}(X)-1 & \text{for } (\bar t,\bar c)=(t_{k+1},c_k) \text{ for any } k\in\{i,i+1,\dots,j-1\};\\ \xi^{t}_{c}(X) & \text{otherwise.}\end{cases}$$

We will show $\tilde X \in M_1$. To do so, by construction of $\tilde X$, first note that $\sum_{t\in T}\xi^t_{\bar c}(\bar X) \le \sum_{t\in T}\xi^t_{\bar c}(X)+1 \le q_{\bar c}$ for any $\bar c\in\{c_i,\dots,c_{j-1}\}$ such that $\sum_{t\in T}\xi^t_{\bar c}(X) < q_{\bar c}$. Next, by construction of $\tilde X$, $\sum_{t\in T}\xi^t_{\bar c}(\bar X) = \sum_{t\in T}\xi^t_{\bar c}(X) = q_{\bar c}$ for every $\bar c\in\{c_i,\dots,c_{j-1}\}$ such that $\sum_{t\in T}\xi^t_{\bar c}(X) = q_{\bar c}$. Moreover, $\sum_{t\in T}\xi^t_{\bar c}(\bar X) \le \sum_{t\in T}\xi^t_{\bar c}(X) = q_{\bar c}$ for every $\bar c\in\{c_i,\dots,c_{j-1}\}$. Finally, for every $\bar c\in C\setminus\{c_i,\dots,c_{j-1}; c_i,\dots,c_{j-1}\}$, $\sum_{t\in T}\xi^t_{\bar c}(\bar X) = \sum_{t\in T}\xi^t_{\bar c}(X) \le q_{\bar c}$. Thus, all school capacities are satisfied by $\bar X$. Also by construction of $\bar X$, for each $\bar d\in D$, $\sum_{\bar c: d(\bar c)=\bar d}\xi^t_{\bar c}(\bar X) = \sum_{\bar c: d(\bar c)=\bar d}\xi^t_{\bar c}(X) = k_{\bar d}$, so $\bar X$ is rationed. Furthermore, for every $\bar c\in C$ and $\bar t\in T$, $\xi^t_{\bar c}(\bar X) \le \max\{\xi^t_{\bar c}(X); \xi^t_{\bar c}(\hat X)\}$ by construction, so all type-specific ceilings are satisfied. Moreover, by construction of $\bar X$, for each $\bar t\in T$, either $\xi^t_{\bar c}(\bar X) = \xi^t_{\bar c}(X)$ for every $\bar c\in C$ or there exists exactly one pair of schools $\bar c'$ and $\bar c''$ in C such that $\xi^t_{\bar c'}(\bar X) = \xi^t_{\bar c'}(\bar X)+1$, $\xi^t_{\bar c''}(\bar X) = \xi^t_{\bar c''}(\bar X)-1$, and $\xi^t_{\bar c}(\bar X) = \xi^t_{\bar c}(X)$ for every $\bar c\in C\setminus\{\bar c',\bar c''\}$. Thus, $\bar t\in T$, $\sum_{\bar c\in C}\xi^t_{\bar c}(\bar X) = \sum_{\bar c\in C}\xi^t_{\bar c}(X)$ for every $\bar t\in T$. Therefore, $\bar X$ is legitimate.

By construction of $\bar X$, either $\xi^t_{d'}(\bar X) = \xi^t_{d'}(X)$ or $\xi^t_{d'}(\bar X) = \xi^t_{d'}(X)-1$. This implies that $\bar X\in M_1$. Furthermore, $\sum_{t,\bar c}|\xi^t_{\bar c}(\tilde X) - \xi^t_{\bar c}(\hat X)| < \sum_{t,\bar c}|\xi^t_{\bar c}(X) - \xi^t_{\bar c}(\hat X)|$, since while creating the $\xi^t_{\bar c}(\tilde X)$ entries, we add 1 to some entries of X that satisfy $\xi^t_{\bar c}(X) < \xi^t_{\bar c}(\hat X)$ and subtract 1 from some entries of X that satisfy $\xi^t_{\bar c}(X) > \xi^t_{\bar c}(\hat X)$. These lead to a contradiction to the assumption that $X\in M_2$, which completes the proof.

Now we are ready to prove the theorem. The "if" part follows from Lemmas 2 and 3. Specifically, by Lemma 2, SPDA produces a legitimate matching. Therefore, by Lemma 3, we have $\underline{p}^t_d \le \xi^t_d(X) \le \bar q^t_d$ for every $t\in T$ and $d\in D$. For each school district d, hence, the maximum proportion of type-t students that can be admitted is $\bar q^t_d/k_d$ and the minimum proportion of type t students that can be admitted is $\underline{p}^t_d/k_d$. Therefore, the ratio difference of type-t students in any two districts is at most $\max_{d\neq d'}\{\bar q^t_d/k_d - \underline{p}^t_{d'}/k_{d'}\}$. We conclude that the α-diversity policy is achieved when $\bar q^t_d/k_d - \underline{p}^t_{d'}/k_{d'} \le \alpha$ for every t, d, and d' with $d\neq d'$.

The "only if" part of the theorem follows from Lemma 4. Suppose that $\bar q^t_d/k_d - \underline{p}^t_{d'}/k_{d'} > \alpha$ for some t, d, and d' with $d\neq d'$. From Lemma 4, we know the existence of a legitimate matching X such that $\xi^t_d(X) = \bar q^t_d$ and $\xi^t_{d'}(X) = \underline{p}^t_{d'}$. Consider a student preference profile where each student prefers her contract in X the most. Then, since the admissions rules

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Proof of Theorem 5Suzuki et al. (2017) study a setting in which each student is initially endowed with a school and there are no constraints associated with student types, that is, when there is just one type. In that setting, they show that if the distribution is M-convex, then their mechanism, called TTC-M, satis es the policy goal, constrained e ciency, individual rationality, and strategy-proofness. To adapt their result to our setting, consider the

To show strategy-proofness, in the original problem, let s be a student, t her type, P_{-s} the preference profile of students other than student s, P_s the true preference of student s, and P_s' a misreported preference of student s. Furthermore, let c and c' be schools assigned to student s under $(P_s; P_{-s})$ and $(P_s'; P_{-s})$ for TTC, respectively. Note that the previous argument establishes that, in the hypothetical problem, student s is allocated to $(c;t)$ and $(c';t)$ under $(\bar{P}_s; \bar{P}_{-s})$ and $(\bar{P}_s'; \bar{P}'_{-s})$, respectively. Because TTC-M is strategy-proof, it follows that $(c;t)\ \bar{P}_s\ (c';t)$ or $c = c'$. By the construction of $\bar{P}_s$,,uction of

Proof of Theorem 7. Let f ( ) = 1 when 2 \ ^{0} and f ( ) = 0 otherwise.

Case 2: Second, consider the case in which there exists no type t' such that $_c^{t'} < \tilde{}_c^{t'}$. Then, $_c^{t'}$ $\tilde{}_c^{t'}$ for every $t' \neq t$. In particular, the total number of students assigned to school c at is strictly larger than at $\tilde{}$. Because everyone is matched with some school at and $\tilde{}$ by assumption, it is implied that, without loss of generality, there exists school $c' \neq c$ such that the total number of students matched with c' is strictly larger at $\tilde{}$ than at . In addition, there exists type t' such that $\tilde{}_{c'}^{t'} > {}_{c'}^{t'}$.

Now we proceed to show condition (1) for this case. To do so, we rst note that $_{c;t} + {}_{c';t'}$ assigns the same number of students as in , so all students are assigned in $_{c;t} + {}_{c';t'}$. Furthermore, it assigns a smaller number of students at school c than , so the capacity constraint at school c is satis ed at $_{c;t} + {}_{c';t'}$. Likewise, $_{c;t} + {}_{c';t'}$ assigns a weakly smaller number of students at c' than $\tilde{}$ does, so the capacity constraint at school c' is satis ed at $_{c;t} + {}_{c';t'}$.

Next we check that the oor for type t and ceiling for type t' at school c are satis ed at $_{c;t} + {}_{c';t'}$. Because $_c^t$ 1 $\tilde{}_c^t$ p_c^t (the rst inequality follows from the assumption $_c^t > \tilde{}_c^t$ and the second from the fact that $\tilde{}$ 2), the oor for type t at school c is satis ed at $_{c;t} + {}_{c';t'}$. Since q_c^t $_c^t > {}_c^t$ 1 (because 2), the ceiling for type t at school c is satis ed at $_{c;t} + {}_{c';t'}$.

Now we check that the oor for type t' and ceiling for type t at school c' are satis ed at $_{c;t} + {}_{c';t'}$. For type t' at school c', we have $_{c'}^{t'} + 1 > {}_{c'}^{t'}$ $p_{c'}^{t'}$ (the rst inequality is obvious and the second follows from the fact that 2), so the oor for type t' at school c' is satis ed for $_{c;t} + {}_{c';t'}$. Furthermore, we have $q_{c'}^{t'}$ $\tilde{}_{c'}^{t'}$ $_{c'}^{t'} + 1$ (the rst inequality follows from the fact that $\tilde{}$ 2 and the second one follows from $\tilde{}_{c'}^{t'} > {}_{c'}^{t'}$), so the ceiling for type t' at school c' is satis ed at $_{c;t} + {}_{c';t'}$.

No other coe cients changed between and $_{c;t} + {}_{c';t'}$, so all other constraints are satis ed at the latter distribution. Therefore, (1) is satis ed.

The proof that (1) is satis ed follows from the facts that $_c^t > \tilde{}_c^t$, $\tilde{}_{c'}^{t'} > {}_{c'}^{t'}$, there are more students assigned to school c at than $\tilde{}$, and there are more students assigned to school c' at $\tilde{}$ than . If we change the roles of with $\tilde{}$, c with c', and t with t', then (1) would imply $\tilde{}$ $_{c';t'} + {}_{c;t}$ 2 $\backslash$ $'$. But this is exactly (2), so we are done. Therefore, $\backslash$ $'$ is an M-convex set.

The desired conclusion then follows from Theorem 5.

Proof of Corollary 3. We show that f is pseudo M-concave. Let $; \tilde{}$ 2 $'$ be distinct. Then U f

$V_1 \quad \{(c', t') \mid \hat{\ }^{c'}_{t'} \quad \tilde{\ }^{c'}_{t'} > \ {}^{c'}_{t'}\}$, $V_2 \quad \{(c', t') \mid \tilde{\ }^{c'}_{t'} > \hat{\ }^{c'}_{t'} > \ {}^{c'}_{t'}\}$, and $V_3 \quad \{(c', t') \mid \tilde{\ }^{c'}_{t'} > \ {}^{c'}_{t'} \quad \hat{\ }^{c'}_{t'}\}$. We consider several cases.

Proof of Corollary 4. Let the balanced-exchange policy be denoted by . We show that $\cap$ $^0 = \{ | \forall d \sum_t {}^t_d = k_d$ and $\forall c\ q_c \geq \sum_t {}^t_c\}$ is M-convex.

Suppose that there exist $, \tilde{} \in \ \cap\ ^0$ such that ${}^t_c > \tilde{}^t_c$. To show M-convexity, we need to find school c' and type t' with ${}^{t'}_{c'} < \tilde{}^{t'}_{c'}$ such that (1) $\ \ _{c,t} + \ _{c',t'} \in \ \cap\ ^0$ and (2) $\tilde{} + \ _{c,t}\ \ _{c',t'} \in \ \cap\ ^0$.

If there exists t' such that $\tilde{}^{t'}_c > {}^{t'}_c$, then the number of students in each district and each school are the same in with $\ \ _{c,t} + \ _{c,t'}$ and $\tilde{}$ with $\tilde{} + \ _{c,t}\ \ _{c,t'}$, so both (1) and (2) are satisfied.

Otherwise, suppose that, for every type $t' \neq t$, $\tilde{}^{t'}_c\ \ {}^{t'}_c$. Therefore, there are more students assigned to school c at than $\tilde{}$. Since the number of students assigned to district d $d(c)$ in and $\tilde{}$ are the same, there exists another school c' in district d such that c' has more students in $\tilde{}$ than . Furthermore, there exists type t' such that $\tilde{}^{t'}_{c'} > {}^{t'}_{c'}$.

We first show (1). Since both schools c and c' are in district d, the number of students assigned to district d is the same at and $\ \ _{c,t} + \ _{c',t'}$. Therefore, the number of students assigned to district d at $\ \ _{c,t} + \ _{c',t'}$ is k_d.

Next we check the school capacity constraints. The number of students assigned to school c at $\ \ _{c,t} + \ _{c',t'}$ is one less than the corresponding number at , so the capacity constraint of school c at $\ \ _{c,t} + \ _{c',t'}$ is satisfied. Furthermore, the number of students assigned to school c' at $\ \ _{c,t} + \ _{c',t'}$ is weakly smaller than the corresponding number at $\tilde{}$. Therefore, the capacity constraint of school c' at $\ \ _{c,t} + \ _{c',t'}$ is also satisfied.

Since all the other coefficients are the same at and $\ \ _{c,t} + \ _{c',t'}$, (1) holds.

Note that the above argument relies on the facts ${}^t_c > \tilde{}^t_c$, ${}^{t'}_{c'} < \tilde{}^{t'}_{c'}$, and $d(c) = d(c')$. If we switch the roles of c with c' and with $\tilde{}$, the implication of (1) is $\tilde{}\ \ _{c',t'} + \ _{c,t} \in \ \cap\ ^0$, which is exactly (2). Therefore, is M-convex.

The result then follows from Theorem 5 because $\cap$ ^{0} is M-convex and the initial matching trivially satisfies the balanced-exchange policy.

Proof of Corollary 5. We first show that the set of distributions $\ \cap\ ^0 = \{ | \forall c, t\ q^t_c\ \ {}^t_c\ \ p^t_c;\ \forall c\ q_c \geq \sum_t {}^t_c$ and $\forall d \sum_t {}^t_d = k_d\}$ is an M-convex set.

Suppose that there exist $, \tilde{} \in \ \cap\ ^0$ such that ${}^t_c > \tilde{}^t_c$. To show M-convexity, we need to find school c' and type t' with ${}^{t'}_{c'} < \tilde{}^{t'}_{c'}$ such that (1) $\ \ _{c,t} + \ _{c',t'} \in \ \cap\ ^0$ and (2) $\tilde{} + \ _{c,t}\ \ _{c',t'} \in \ \cap\ ^0$. Let d $d(c)$. To show both conditions, we look at two possible cases depending on whether $c' = c$ or not.

Case 1: First consider the case when there exists type t' such that ${}^{t'}_c < \tilde{}^{t'}_c$. We prove (1) that $\ \ _{c,t} + \ _{c,t'} \in \ \cap\ ^0$. Since $\ \ _{c,t} + \ _{c,t'}$ assigns the same total number of students at school c as , the capacity constraint at school c at $\ \ _{c,t} + \ _{c,t'}$ is satisfied. Furthermore, the number of students assigned to any district at $\ \ _{c,t} + \ _{c,t'}$ is the same as , which means that the number of students in every district is equal to the number of students who are from there. Next, because $^{t}_{c}$ 1 $\tilde{}^{t}_{c}$ p^{t}_{c} (the former inequality comes from the assumption $^{t}_{c} > \tilde{}^{t}_{c}$, and the latter comes from the fact $\tilde{}$ 2 \ ^{0}), the oor for type t at school c is satis ed at $_{c;t}$ + $_{c;t^0}$. Next, the facts that 2 \ ^{0} and $^{t}_{c} > \tilde{}^{t}_{c}$ implies q^{t}_{c} $^{t}_{c}$ $\tilde{}^{t}_{c}$ + 1. Therefore, the ceiling for type t at school c at $_{c;t}$ + $_{c;t^0}$ is satis ed.

The oor for type t^0 at school c is satis ed for $_{c;t}$ + $_{c;t^0}$ because $^{t^0}_{c}$ + 1 $^{t^0}_{c}$ $p^{t^0}_{c}$ (the former inequality is obvious, and the latter comes from the fact 2 \ ^{0}). Similarly, the ceiling for type t^0 at school c is satis ed at $_{c;t}$ + $_{c;t^0}$ because q^{t}_{c} $\tilde{}^{t^0}_{c}$ $^{t^0}_{c}$ + 1.

No other coe cients changed between and $_{c;t}$ + $_{c;t^0}$, so all other constraints are satis ed at the latter distribution. Therefore, (1) is satis ed.

The proof that (1) is satis ed follows from the facts that $^{t}_{c} > \tilde{}^{t}_{c}$ and $^{t^0}_{c} < \tilde{}^{t^0}_{c}$. By changing the roles of t with t^0 and with $\tilde{}$ in the preceding argument, we get the implication of (1) that $\tilde{}$ $_{c;t^0}$ + $_{c;t}$ 2 \ ^{0}. But this is exactly (2).

Case 2: In this case, $c^0 \neq c$ for every (c^0, t^0) such that $^{t^0}_{c^0} < \tilde{}^{t^0}_{c^0}$. Then, $^{t^0}_{c}$ $\tilde{}^{t^0}_{c}$ for every $t^0 \neq t$. In particular, the total number of students assigned to school c at is strictly larger than at $\tilde{}$. Because the number of students in district d are the same at and $\tilde{}$, there exist school c^0 in district d such that the total number of students matched with c^0 is strictly larger at $\tilde{}$ than at . In addition, there exists type t^0 such that $\tilde{}^{t^0}_{c^0} > {}^{t^0}_{c^0}$.

Now we proceed to show condition (1) for this case. To do so, we rst note that $_{c;t}$ + $_{c^0;t^0}$ assigns the same number of students to each district as in , so the number of students assigned to each district d is k_d

No other coefficients changed between and $_{c;t} + _{c^0;t^0}$, so all other constraints are satisfied at the latter distribution.

The proof that (1) is satisfied follows from the facts that $d(c) = d(c^0)$, $^t_c > \tilde{}^t_c$, $\tilde{}^{t^0}_{c^0} > ^{t^0}_{c^0}$, there are more students assigned to school c at than $\tilde{}$, and there are more students assigned to school c^0 at $\tilde{}$ than . If we change the roles of with $\tilde{}$, c with c^0, and t with t^0, then (1) would imply $\tilde{}$ $_{c^0;t^0} + _{c;t} \in \cap ^0$. But this is exactly (2), so we are done.

The result then follows from Theorem 5 because $\cap ^0$ is M-convex.

Proof of Theorem 8. Suppose that district admissions rules favor their own students. Fix a student preference profile. Recall that under interdistrict school choice, students are assigned to schools by SPDA, where each student ranks all contracts associated with her and each district d has the admissions rule Ch_d. Under intradistrict school choice, students are assigned to schools by SPDA where students only rank the contracts associated with their home districts and each district d has the admissions rule Ch_d. We first show that the intradistrict SPDA outcome can be produced by SPDA when all districts participate simultaneously and students rank all contracts, including the ones associated with the other districts, by modifying admissions rules for the districts. Let $Ch^0_d(X) \quad Ch_d(\{x \in X \mid d(s(x)) = d\})$ be the modified admissions rule.

In SPDA, if district admissions rules have completions that satisfy path independence, then SPDA outcomes are the same under the completions and the original admissions rules because in SPDA a district always considers a set of proposals which is feasible for students. Furthermore, SPDA does not depend on the order of proposals when district admissions rules are path independent. As a result, SPDA does not depend on the order of proposals when district admissions rules have completions that satisfy path independence. Therefore, the intradistrict SPDA outcome can be produced by SPDA when all districts participate simultaneously and students rank all contracts including the ones associated with the other districts and each district d has the admissions rule Ch^0_d. The reason behind this is that when each district d has admissions rule Ch^0_d, a student is not admitted to a school district other than her home district. Furthermore, because Ch_d favors own students, the set of chosen students under Ch^0_d is the same as that under Ch_d for any set of contracts of the form $\{x \in X \mid d(s(x)) = d\}$ for any set X.

We next show that Ch^0_d has a path-independent completion. By assumption, for every district d, there exists a path-independent completion $\widetilde{Ch}_d$ of Ch_d. Let $\widetilde{Ch}^0_d(X) \quad \widetilde{Ch}_d(\{x \in X \mid d(s(x)) = d\})$. We show that $\widetilde{Ch}^0_d$ is a path-independent completion of Ch^0_d. To show that $\widetilde{Ch}^0_d(X)$ is a completion, consider a set X such that $\widetilde{Ch}^0_d(X)$ is feasible for students. Let $X \quad \{x \in X \mid d(s(x)) = d\}$. Then we have the following:

$$\widetilde{Ch}^0_d(X) = \widetilde{Ch}_d(X) = Ch_d(X) = Ch^0_d(X);$$

District d_n accepts $Ch_{d_n}(\mu^{n-1}_{d} \cup X^n_{d_n})$ and rejects the rest of the contracts. Let $\mu^n_{d_n} = Ch_{d_n}(\mu^{n-1}_{d_n} \cup X^n_{d_n})$ and $\mu^n_d = \mu^{n-1}_d \setminus Y^n$ where $Y^n \equiv \{x \in \mu^{n-1} | \exists y \in \mu^n_{d_n}$ s.t. $s(x) = s(y)\}$ for $d \neq d_n$. If there are no blocking contracts for matching μ^n under $(Ch_d)_{d \in D}$, then stop and return μ^n, otherwise go to Step $n+1$.

We show that district d_n does not reject any contract in $\mu^{n-1}_{d_n}$ by mathematical induction on n, i.e., $\mu^n_{d_n} \supseteq \mu^{n-1}_{d_n}$ for every $n \geq 1$. Consider the base case for $n = 1$. Recall that $\mu^1_{d_1} = Ch_{d_1}(\mu^0_{d_1} \cup X^1_{d_1}) = Ch_{d_1}(\mu^0_{d_1} \cup X^1_{d_1})$. By construction, $\mu^1_{d_1}$ is a feasible matching. We claim that $\mu^0_{d_1} \cup \mu^1_{d_1}$ is feasible for students. Suppose, for contradiction, that it is not feasible for students. Then there exists a student s who has one contract in $\mu^0_{d_1}$ and one in $\mu^1_{d_1} \setminus \mu^0_{d_1}$. Call the latter contract z. By construction, $z \, P_s \, \mu^0_s$, and by path independence, $z \in Ch_{d_1}(\mu^0_{d_1} \cup \{z\})$. Furthermore, since student s is matched with district d_1 in μ^0, $d(s) = d_1$. Therefore, $Ch_{d_1}(\mu^0_{d_1} \cup \{z\}) = Ch^0_{d_1}(\mu^0_{d_1} \cup \{z\})$ by definition of $Ch^0_{d_1}$ and construction of μ^0. Hence, $z \in Ch^0_{d_1}(\mu^0_{d_1} \cup \{z\})$, which contradicts the fact that μ^0 is stable under $(Ch^0_d)_{d \in D}$. Hence, $\mu^0_{d_1} \cup \mu^1_{d_1}$ is feasible for students. Feasibility for students implies that $Ch_{d_1}(\mu^0_{d_1} \cup \mu^1_{d_1}) \subseteq Ch^0_{d_1}(\mu^0_{d_1} \cup \mu^1_{d_1})$. Path independence and construction of $\mu^1_{d_1}$ yield $\mu^1_{d_1} = Ch_{d_1}(\mu^0_{d_1} \cup \mu^1_{d_1})$. Furthermore, there exists no student s, such that $d(s) = d_1$, who has a contract in $\mu^1_{d_1} \setminus \mu^0$, as this would contradict stability of μ^0 under $(Ch^0_d)_{d \in D}$. This implies, by definition of $Ch^0_{d_1}$, that $Ch^0_{d_1}(\mu^0_{d_1} \cup \mu^1_{d_1}) = Ch^0_{d_1}(\mu^0_{d_1})$, and, by stability of μ^0 under $(Ch^0_d)_{d \in D}$, $Ch^0_{d_1}(\mu^0_{d_1}) = \mu^0_{d_1}$. Therefore, $\mu^1_{d_1} = Ch_{d_1}(\mu^0_{d_1} \cup \mu^1_{d_1}) \subseteq Ch^0_{d_1}(\mu^0_{d_1} \cup \mu^1_{d_1}) = \mu^0_{d_1} = \mu^0_{d_1}$, which means that district d_1 does not reject any contracts.

Now consider district d_n where $n > 1$. There are two cases to consider. First consider the case when $d_n \neq d_i$ for every $i < n$. In this case, $\mu^{n-1}_{d_n} \subseteq \mu^0_{d_n} = \mu^0_{d_n}$. We repeat

Therefore, district d_n gets at least one new contract at Step n. Hence, at least one student gets a strictly more preferred contract at every step of the algorithm while every other student gets a weakly more preferred contract. Since the number of contracts is finite, the algorithm has to end in a finite number of steps.

Because the interdistrict SPDA outcome under $(Ch_d)_{d\in D}$ is the same as the interdistrict SPDA outcome under $(\tilde{Ch}_d)_{d\in D}$ and the interdistrict SPDA outcome under $(Ch^0_d)_{d\in D}$ is the same as the interdistrict SPDA outcome under $(\tilde{Ch}^0_d)_{d\in D}$, the lemma implies that every student weakly prefers the outcome of interdistrict SPDA under $(Ch_d)_{d\in D}$ to the outcome of intradistrict SPDA (which is the same as the interdistrict SPDA outcome under $(Ch^0_d)_{d\in D}$). This completes the proof of the first part.

To prove the second part of the theorem, we show that if at least one district's admissions rule fails to favor own students, then there exists a preference profile such that not every student is weakly better off under interdistrict SPDA. Suppose that for some district d, there exists a matching X, which is feasible for students, such that $Ch_d(X)$ is not a superset of $Ch_d(X_d)$, where $X_d \equiv \{x \in X \mid d(s(x)) = d\}$. Now, consider a matching Y where (i) all students from district d are matched with schools in district d, (ii) Y is feasible, and (iii) $Y \supseteq Ch_d(X_d)$. The existence of such a Y follows from the fact that $Ch_d(X_d)$ is feasible and $k_{d^0} \le \sum_{c:d(c)=d^0} q_c$, for every district d^0 (that is, there are enough seats in district d^0 to match all students from district d^0

In interdistrict SPDA, at the rst step, each student who has a contract in X proposes that contract and every other student proposes a contract associated with a district di erent from d. District d considers X (or X_d), and tentatively accepts $Ch_d(X)$. Because $Ch_d(X)$ 6 $Ch_d(X)$ by assumption, at least one student who has a contract in $Ch_d(X)$ is rejected. Therefore, this student is strictly worse o under interdistrict school choice.

Proof of Theorem 9. To show the result, we rst introduce the following weakening of the substitutability condition (Hat eld and Kojima, 2008). A district admissions rule Ch_d satis es *weak substitutability* if, for every x 2 X Y X with x 2 $Ch_d(Y)$ and jY_sj 1 for each s 2 S, it must be that x 2 $Ch_d(X)$.

Under weak substitutability, the following result is known (the statement is slightly modi ed for the present setting).

Theorem 10 (Hat eld and Kojima (2008)). Let d and d^0 be two distinct districts. Suppose that Ch_d satis es IRC but violates weak substitutability. Then, there exist student preferences and a path-independent admissions rule for d^0 such that, regardless of the other districts' admissions rules, no stable matching exists.

Given this result, for our purposes it su ces to show the following.

Theorem 9'. Let d be a district. There exist a set of students, their types, schools in d, and type-speci c ceilings for d such that there is no district admissions rule of d that has district-level type-speci c ceilings, is d-weakly acceptant, and satis es IRC and weak substitutability.

To show this result, consider a district d with $k_d = 2$. There are three schools c_1, c_2, c_3 in the district, each with capacity one, and four students s_1, s_2, s_3, s_4 of which two are from a di erent district. Students s_1 and s_2

cases that satisfy d-weak acceptance and type-speci c ceilings are $f(s_2;c_2)g$ and $f(s_1;c_1);(s_3;c_2)g$. The latter would violate weak substitutability since in that case $(s_3;c_2)$ would be accepted in a larger set $f(s_1;c_1);(s_2;c_2);(s_3;c_2)g$ and rejected from a smaller set $f(s_2;c_2);(s_3;c_2)g$. Then, by IRC, $Ch_d(f(s_1;c_1);(s_2;c_2);(s_3;c_2)g) = f(s_2;c_2)g$ implies $Ch_d(f(s_1;c_1);(s_2;c_2)g) = f(s_2;c_2)g$. Then we note that $Ch_d(f(s_1;c_1);(s_2;c_2);(s_3;c_1)g) = f(s_2;c_2);(s_3;c_1)g$ since by weak substitutability $(s_1;c_1)$ cannot be chosen, and therefore $(s_2;c_2)$ and $(s_3;c_1)$ have to be chosen due to d-weak acceptance. Next, again by weak substitutability, we note that $Ch_d(f(s_1;c_1);(s_2;c_2);(s_3;c_1)g) = f(s_2;c_2);(s_3;c_1)g$ implies $Ch_d(f(s_1;c_1);(s_3;c_1)g) = f(s_3;c_1)g$. Finally, we note that this contradicts with $Ch_d(f(s_1;c_1);(s_2;c_1);(s_3;c_1);(s_4;c_1)g) = f(s_1;c_1)g$ and IRC.

(2) Suppose $Ch_d(f(s_2;c_2);(s_3;c_2);(s_4;c_2)g) = f(s_3;c_2)g$. Consider $Ch_d(f(s_2;c_3);(s_4;c_3)g)$. This can be either $f(s_2;c_3)g$ or $f(s_4;c_3)g$. We consider these two possible cases separately. These two subcases will follow similar arguments to Case (1) above and change the indexes appropriately in order to get a contradiction.

(a) Suppose $Ch_d(f(s_2;c_3);(s_4;c_3)g) = f(s_2;c_3)g$. Next, we argue that $Ch_d(f(s_1;c_1);(s_2;c_3);(s_4;c_3)g) = f(s_2;c_3)g$. This is because the only two cases that satisfy d-weak acceptance and type-speci c ceilings are $f(s_2;c_3)g$ and $f(s_1;c_1);(s_4;c_3)g$. The latter would violate weak substitutability since in that case $(s_4;c_3)$ would be accepted in a larger set $f(s_1;c_1);(s_2;c_3);(s_4;c_3)g$ and rejected from a smaller set $f(s_2;c_3);(s_4;c_3)g$. Then, by IRC, $Ch_d(f(s_1;c_1);(s_2;c_3);(s_4;c_3)g) = f(s_2;c_3)g$ implies $Ch_d(f(s_1;c_1);(s_2;c_3)g) = f(s_2;c_3)g$. Then we note that $Ch_d(f(s_1;c_1);(s_2;c_3);(s_4;c_1)g) = f(s_2;c_3);(s_4;c_1)g$ since by weak substitutability $(s_1;c_1)$ cannot to be chosen, therefore $(s_2;c_3)$ and $(s_4;c_1)$ have to be chosen due to d-weak acceptance. Next, again by weak substitutability, we note that $Ch_d(f(s_1;c_1);(s_2;c_3);(s_4;c_1)g) = f(s_2;c_3);(s_4;c_1)g$ implies $Ch_d(f(s_1;c_1);(s_4;c_1)g) = f(s_4;c_1)g$. Finally, we note that this contradicts with $Ch_d(f(s_1;c_1);(s_2;c_1);(s_3;c_1);(s_4;c_1)g) = f(s_1;c_1)g$ and IRC.

(b) Suppose $Ch_d(f(s_2;c_3);(s_4;c_3)g) = f(s_4;c_3)g$. Next, we argue that $Ch_d(f(s_2;c_3);(s_3;c_2);(s_4;c_3)g) = f(s_4;c_3)g$. This is because the only two cases that satisfy d-weak acceptance and type-speci c ceilings are $f(s_4;c_3)g$ and $f(s_2;c_3);(s_3;c_2)g$. The latter would violate weak substitutability since in that case $(s_2;c_3)$ would be accepted in a larger set $f(s_2;c_3);(s_3;c_2);(s_4;c_3)g$ and rejected from a smaller set $f(s_2;c_3);(s_4;c_3)g$. Then, by IRC, $Ch_d(f(s_2;c_3);(s_3;c_2);(s_4;c_3)g) = f(s_4;c_3)g$ implies $Ch_d(f(s_3;c_2);(s_4;c_3)g) = f(s_4;c_3)g$. Then we note that $Ch_d(f(s_2;c_2);(s_3;c_2);(s_4;c_3)g) = f(s_2;c_2);(s_4;c_3)g$

since by weak substitutability $(s_3; c_2)$ cannot to be chosen, therefore $(s_4; c_3)$ and $(s_2; c_2)$ have to be chosen due to d-weak acceptance. Next, again by weak substitutability, we note that $Ch_d(f(s_2; c_2); (s_3; c_2); (s_4; c_3)g) = f(s_2; c_2); (s_4; c_3)g$ implies $Ch_d(f(s_2; c_2); (s_3; c_2)g) = f(s_2; c_2)g$. Finally, we note that this contradicts with $Ch_d(f(s_2; c_2); (s_3; c_2); (s_4; c_2)g) = f(s_3; c_2)g$ and IRC.