In this paper, imposing a uniform upper bound in coalition sizes, we provide a direct proof of nonemptiness of the core using Kakutani's fixed point theorem so that their important theorem is easily accessible to more application-oriented researchers.

Moreover, this direct proof allows us to drop the comprehensiveness assumption entirely for nonemptiness of the f-core. This generalization broadens the applicability of our nonemptiness result of the f-core to a significantly wider class of problems such as matching problems in large markets: for example, couples or more generally, preferences over colleagues in one-to-many matching problems, hedonic games, and network formation problems when the size of each component's diameter is bounded above by a finite number.[2] Interestingly, in these cases, the equal treatment (in payoffs) property for players of the same type can be violated in every f-core allocation.

Our results are applicable to the models in the literature of matchings with atomless players such as Legros and Newman (1996), Konishi (2013), Gersbach, and Haller, and Konishi (2015), and Chade and Eeckhout (2020) as well as to atomless versions of the standard matching and hedonic problems such as Alkan (1986), Dutta and Masso (1997), Konishi, Quint, and Wako (2001), Banerjee, Konishi, and Sonmez (2001), and Bogomolnaia and Jackson (2002). We also discuss applications of our results to Scarf's (1971) nonemptiness result for the core of NTU games. We can also relate our results with the ones in Konishi, Pan, and Simeonov (2023) that analyze a team competition problem in a large market in the presence of moral hazard, showing the existence of a free-entry equilibrium of a team formation game.

The rest of the paper is organized as follows. Section 2 introduces a simplified version of the Kaneko-Wooders model and assumptions. Section 3 presents an atomless player version of a popular roommate example in a one-sided matching problem, and discusses how the f-core looks like in this example. Section 4 proves our main theorems. Theorem 1 shows that with the comprehensiveness condition, the equal-treatment f-core is nonempty. In contrast, our main result, Theorem 2, proves the nonemptiness of f-core without comprehensiveness, but there can be player types who are treated unequally in an f-core allocation.

2 The Model

There is a set of player types T, each of which has a continuum of atomless players of measure $_t > 0$ for each $t \in T$. Each coalition type is described by its membership profile, $(m^{\alpha}_t)_{t \in T}$, where $m^{\alpha}_t \in \mathbb{Z}_+$ is the number of type t players in coalition . Let the set of all admissible coalition types be .

Let $T^{\alpha} = \{t \in T : m^{\alpha}_t > 0\}$. For each coalition type $_{t \in T}$

Our NTU model G is summarized by a list $G = (T; \Gamma; (\ \ _t)_{t \in T}; (V^\gamma)_{\gamma \in \Gamma})$. Let $\Gamma^t \equiv \{\gamma \in \Gamma : m^\gamma_{t'} = 0 \ \forall t' \neq t\}$, which is the set of coalition types that consists of only type t players. For $\gamma \in \Gamma^t$, let $\underline{u}^\gamma_t$ be the smallest upper bound such that $\underline{u}^\gamma_t \geq u_t$ for all $u_t \in V^\gamma$. For each $t \in T$, let $\underline{u}_t \equiv \max_{\gamma \in \Gamma^t} \underline{u}^\gamma_t$, this is the payoff guaranteed for type t player in a core allocation (individual rationality). Without loss of generality, for all $\gamma \in \Gamma$, we translate V^γ so that the individually rational payoff for type t is positive: i.e. $\underline{u}_t > 0$ for all $t \in T$.

(A1) T is a finite set

(A2) $V^\gamma - \mathbb{R}^{T^\gamma}_+ = V^\gamma$ for all $\gamma \in \Gamma$ (Comprehensiveness)

(A3) $V^\gamma \cap \mathbb{R}^{T^\gamma}_+$ is compact

(A4) Measure Consistency

(A5) There is $K \in \mathbb{Z}_{++}$ such that for all $\gamma \in \Gamma$, $0 < \sum_{t \in T} m^\gamma_t \leq K$ holds.

Assumptions (A1)-(A4) are employed in Kaneko and Wooders (1986). For the last technical condition (A4), see Kaneko and Wooders (1986)[4].

Our only simplification assumption of this paper is (A5): Kaneko and Wooders (1986) assume a weaker assumption, *per capita boundedness*. Note that (A1) is

The **equal-treatment f-core** for G is a collection of all equal-treatment f-core allocations. Clearly, an equal-treatment f-core allocation for G is an f-core allocation for G as well.

The results of this paper are as follows:

Theorem 1. The equal-treatment f-core is nonempty under (A1), (A2), (A3), (A4), and (A5).

Theorem 2. The f-core is nonempty under (A1), (A3), (A4), and (A5).

Theorem 3. The f-core and the equal-treatment f-core are equivalent under (A1), (A2'), (A3), (A4), and (A5).

The differences between these theorems come from the assumptions around (A2), "Comprehensiveness." Although the main theorem is Theorem 2, the same type players might get different payoffs in every f-core allocation. In the following section, we present two simple educational examples, providing detailed analyses.

3 Examples

Here, we present two examples to illustrate our results before we present formal proofs. First, consider a continuum version of a roommate example in a hedonic game (Banerjee et al. 2001).

Example 1. Let $T = \{1; 2; 3\}$ and $K = 2$. There are only 6 feasible coalitions, and players' payoff vector in each coalition is determined uniquely (hedonic game): $(u_1; u_2) = (3; 2)$ for coalition $\{1; 2\}$

for all $t = 1, 2, 3$. There are coalitions $\{1, 2\}$, $\{2, 3\}$, and $\{3, 1\}$ with measure $\frac{1}{2}$ each, and each coalition o ers (weakly suboptimal) payo $(2, 2)$ for its members. Note that there is no strictly improving coalitional deviation. It is because coalition $\{1, 2\}$ improves type 1 player's payo from 2 to 3, while type 2 player's payo is unchanged. In our setting, there is no means to transfer utility across players in the same coalition (unlike (A2')), and thus there is no possible deviations from weakly Pareto-inferior allocation. Symmetrically, there is no possibility for any coalitional deviation to improve all players in the coalition. This equal-treatment f-core allocation is shown in Figure 1.

Figure 1: *The f-core allocations from Example 1 with and without comprehensiveness.*

Now, suppose that (A2) is dropped. Then, the above payo vector is no longer feasible: $(u_1, u_2) = (2, 2) \notin V^{\{1,2\}} = \{(3, 2)\}$. Thus, there is no equal treatment f-core allocation. However, there is a weakly Pareto-improving payo vector in the original hedonic game: $(u_1, u_2) = (3, 2) \in V^{\{1,2\}}$. Since payo vector $(u_1, u_2, u_3) = (2, 2, 2)$ cannot be blocked by any nite coalitions, $(u_1, u_2) = (3, 2)$ cannot be blocked either. Thus, we have an f-core of the original hedonic game: ${}_1^{\{1,2\}} = {}_2^{\{1,2\}} = {}_2^{\{2,3\}} = {}_3^{\{2,3\}} = {}_3^{\{3,1\}} = {}_1^{\{3,1\}} = \frac{1}{2}$, $(u_1^{\{1,2\}}, u_2^{\{1,2\}}) = (3, 2)$, $(u_2^{\{2,3\}}, u_3^{\{2,3\}}) = (3, 2)$, and $(u_3^{\{3,1\}}, u_1^{\{3,1\}}) = (3, 2)$. In this allocation, also shown in Figure 1, one half of each type players are getting less payo than the other half. Despite of this apparent unequal treatment, there is no way for the worse-o players to form a strictly improving coalitional

points (i) $(u_1, u_2) = (3, 2)$ and (ii) $(u_1, u_2) = (2, 3)$ is an f-core allocation. In this case, (A2) is satisfied, but (A2') is violated. If (A2') is satisfied, a type 2 player in (i) can approach to a type 1 player in (ii) offering a payoff $u_1^0 \in (2, 3)$. Then, by (A2'), this type 2 player can obtain a payoff higher than 2.

4 Proofs of the Theorems

The main theorem of this paper is Theorem 2, but we utilize Theorem 1 in order to prove it. We will illustrate how the proof of Theorem 1 is constructed by using Example 1. Starting with the original hedonic game, we take comprehensive covers of the original payoff vectors: $V^{\{1,2\}} = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 \leq 3, u_2 \leq 2\}$, $V^{\{2,3\}} = \{(u_2, u_3) \in \mathbb{R}^2 : u_2 \leq 3, u_3 \leq 2\}$, $V^{\{3,1\}} = \{(u_3, u_1) \in \mathbb{R}^2 : u_3 \leq 3, u_1 \leq 2\}$, and $V^{\{t\}} = \{u_t \in \mathbb{R} : u_t \leq 1\}$ for all $t = 1, 2, 3$. For each two person coalition, we now take the weak Pareto efficient set $\partial V^{\{t,t+1\}} = \{(u_t, u_{t+1}) \in \mathbb{R}^2 : u_t \in [0, 3], u_{t+1} = 2\} \cup \{(u_t, u_{t+1}) \in \mathbb{R}^2 : u_t = 3, u_{t+1} \in [0, 2]\}$ for all $t = 1, 2, 3$. Let $^t \equiv \{\{t, t-1\}, \{t, t+1\}, \{t\}\}$ be the set of coalitions that type 8]; u

mation please consult Figure 2 below. Let the inverse function of f^{Y} be u^{Y} : ^{Y} *!* @∇^{Y}. This continuous mapping can be interpreted that $u^{Y}(x^{Y})$ is a payo vector for each abstract policy x^{Y}

$u_t^{y}(x^{y}) = u_t$ for all t 2 T. Thus, type t players get u_t almost everywhere. For all 2 ~, we have $\frac{\nu_{t^0}^{y}}{m_{t^0}^{y}} = \frac{\nu_t^{y}}{m_t^{y}}$ for all $t; t^0$ 2 T^{y}

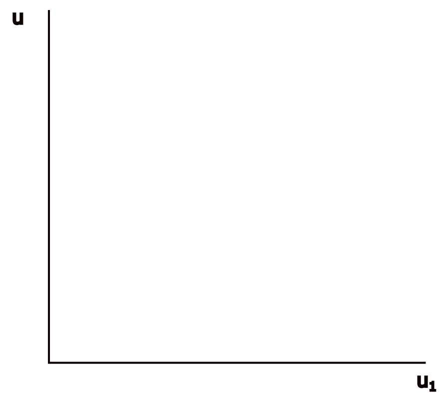

Figure 3: *The comprehensive hull and its weak-Pareto frontier (Theorem 2).*

Theorem 3. The f-core and the equal-treatment f-core are equivalent under (A1), (A2'), (A3), (A4), and (A5).

Proof of Theorem 3. First note that under (A2'), in any coalition 2 with $^{\gamma}_{t} > 0$ for t 2 T^{γ}, two distinct allocations u^{γ} and $\tilde{u}^{\gamma}$ cannot coexist in an f-core allocation. Suppose not. Then there exist 2 and $u^{\gamma}; \tilde{u}^{\gamma}$ 2 V^{γ}

holds for some with $^{y}_{t} > 0$ with a strict inequality for at least one t 2 T^{y}.
The strict f-core for

$(T, (V(S))_{S \subseteq T, S \neq \emptyset})$ is **Scarf-balanced** if every balanced subfamily B of 2^T, it follows that $\cap_{S \in B} V(S) \subseteq V(T)$. Scarf's theorem (1971) is as follows.

Corollary 2 (Scarf, 1971). Let $(T, (V(S))_{S \subseteq T, S \neq \emptyset})$ be an NTU game. The core of an NTU game $(T, (V(S))_{S \subseteq T, S \neq \emptyset})$ is nonempty if

(B1) $V(S) - \mathbb{R}^T_+ = V(S)$ for all $S \subseteq T, S \neq \emptyset$ (Comprehensiveness)

(B2) $V^S \cap \mathbb{R}^S_+$ is compact for all $S \subseteq T, S \neq \emptyset$

(B3) $(T, (V(S))_{S \subseteq T, S \neq \emptyset})$ is Scarf-balanced.

Proof. Consider the following special case of our problem in order to connect it with the standard NTU game: $\ _1 = \dots = \ _T = 1$, $\ \equiv \{S \subseteq T : S \neq \emptyset\}$, and $m^S_t = 1$ for all $t \in S \in 2^T \setminus \{\emptyset\} = \ $ (where $T^S \equiv S$ for all $S \in \ $). This special case is an atomless player version of a standard T-person NTU game $(T, (V(S))_{S \subseteq T, S \neq \emptyset})$. By Theorem 1, we have an f-core allocation $((\ ^S_t)_{t \in S, S \in \ }, (u_t)_{t \in T})$. Since $(\ ^S_t)_{t \in S, S \in \ }$ is a feasible assignment, $\sum_{S \subseteq T, S \neq \emptyset} \ ^S_t = 1$ for all $t \in T$ and $\ ^S_t = \ ^S_r$ for all $t, r \in S$. Let $\ ^S = \ ^S_t$ for $t \in S \subseteq T$. This implies that $B \equiv \{S \subseteq T : \ ^S > 0\}$ is a balanced family and $\{\ ^S : S \in B\}$ is an associated balanced coefficients. Since $(u_t)_{t \in S} \in V(S)$ for any $S \in B$, $(u_t)_{t \in T} \in \cap_{S \in B} V(S)$ holds. By Scarf-balancedness, $(u_t)_{t \in T} \in V(T)$. By the definition of an f-core allocation, there is no $S \in \ $ and $(u'_t)_{t \in T^S} \in V^S$ such that $u'_t > u_t$ for all $t \in T^S$. Hence, if an NTU game is Scarf-balanced, there is a core allocation $(u_t)_{t \in T} \in V(T)$.

5.4 Competing Teams and Contracts

Alchian and Demsetz (1972) considered a team production problem in the presence of moral hazard in a partial equilibrium model, and Holmstrom (1982) showed that an efficient allocation is achievable depending on the class of contracts available for teams. We can illustrate how a team formation problem in a competitive environment can be incorporated in our framework to analyze an equilibrium team structure with optimal contracts, allowing for limited freedom for teams to choose their contracts. Let V^γ be a collection of all implementable payoff vectors for all available contracts for team-type $\ $. If the set V^γ is a compact set for all $\ \in \ $, Theorem 2 shows that there is an f-core allocation.[6] That is, the f-core allocation is an allocation in which each team-type $\ $ uses a contract, such that there is no feasible contract that can improve all members' payoffs. That is, an f-core allocation is an equilibrium competing contract structure — a list of team contracts that cannot be shaken by any other contracts by entrants with new contracts (Konishi, Pan, and Simeonov 2023). In addition, we can allow for wide-spread externalities due to market price changes — if

[6] Moral hazard problems may not necessarily have binding individual rationality constraints due to limited liability by the agent, and the comprehensiveness assumption could be violated. In such a case, an f-core allocation may not satisfy the equal-treatment property

market price changes, the set of achievable payoffs V^y can change as well. Hammond, Kaneko, and Wooders (1989) and Kaneko and Wooders (1989) introduced widespread externalities to Kaneko and Wooders model (1986), and showed that the f-core is nonempty using the property that atomless coalitions' deviations do not affect the whole economy (in contrast, the Aumann core can be empty under widespread externalities due to the atomic impact of a large (positive-measure) coalition's deviation). Our fixed-point-based proof strategy turns out to be useful even under widespread externalities as is shown in Konishi, Pan, and Simeonov (2023).

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